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Spectral invariant of Floer homology and its application to Hill's lunar problem

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Spectral invariant of Floer homology and its application to Hill's lunar problem

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Abstract

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In this thesis, we reinterpret spectral invariants in symplectic homology and wrapped Floer homology as symplectic capacities for fiberwise star-shaped domains in a cotangent bundle of a closed orientable manifold. We compute the spectral invariants of various homology classes in symplectic homology and wrapped Floer homology for the fiberwise star-shaped domains defined by the rotating Kepler problem. Moreover we prove inclusions among the fiberwise star-shaped domains defined by the rotating Kepler problem and Hill's lunar problem. Finally if we combine computations of spectral invariants and result of inclusions, then we obtain estimates for spectral invariants in the symplectic homology and the wrapped Floer homology of Hill's lunar problem using monotonicity of spectral invariants. As a result, using spectrality of spectral invariants, these estimates for spectral invariants of Hill's lunar problem give us estimates of the action values of periodic orbits, symmetric periodic orbits and doubly symmetric orbits in Hill's lunar problem. As a Corollary, we can obtain systole bounds for the regularized Hill's lunar problem: For $c > c_H^0$, there is at least one periodic Reeb orbit whose action is less than π on $(\Sigma_H^c, \lambda_{can})$. Moreover, we can say the same result for symmetric periodic Reeb orbits and for doubly symmetric periodic Reeb orbits. Furthermore, we obtain a sequence of intervals which insure the existence of a (symmetric) periodic orbit whose action lies on each of the intervals.

Key words: Hamiltonian Dynamics, The Rotating Kepler Problem, Hill's Lunar Problem, Floer Homology, Spectral Invariant, Fiberwise Convexity
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Contents

Chapter 1

Introduction

Study of the motion of the Moon has been one of most interesting objects in celestial mechanics. Before Hill, the accuracy of the lunar theory was not so good. Hill introduced Hill's lunar problem for the lunar theory which reflects successfully the perturbation effect of the Sun. Hill's lunar problem can be derived from the (circular planar) restricted three body problem. The restricted three body problem is obtained from the three body problem by assuming one particle, say M(Moon), is massless and two primaries, say S, E(Sun, Earth), take the Keplerian circular motion on the plane. [1] With suitable normalization of physical constants, the time-independent Hamiltonian of the restricted three body problem is

$$H_{R3BP} : T^*(\mathbb{R}^2 \setminus \{(-\mu, 0), (1-\mu, 0)\}) \to \mathbb{R},$$
$$H_{R3BP}(q,p) := \frac{1}{2}|p|^2 - \frac{1-\mu}{|q-(\mu,0)|} - \frac{\mu}{|q-(1-\mu,0)|} + p_1 q_2 - p_2 q_1$$

where $\mu = \frac{M_E}{M_S + M_E}$ is the mass ratio between the mass of the Earth and the total mass. We will derive this Hamiltonian and discuss its properties in

[1] In the restricted three body problem, many authors use convention letting the massless particle S(Satellite) and two primaries E, M(Earth, Moon). Here we use the Moon as a massless particle in the Sun-Earth system in order to emphasize the relation with Hill's lunar problem.

section 3.1. If one takes the limit $\mu \to 0$, then we get the Hamiltonian

$$H_{RKP} : T^*(\mathbb{R}^2 \setminus \{(0,0)\}) \to \mathbb{R}, \quad H_{RKP}(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} + p_1q_2 - p_2q_1$$

of the rotating Kepler problem. As one can see, this is the Kepler problem on the rotating reference frame. The rotating Kepler problem is completely integrable. Moreover, we can compute all periodic orbits of the rotating Kepler problem. These properties will be discussed in section 3.2.

Hill's lunar problem is another limit problem of the restricted three body problem. Hill's lunar problem can be obtained by not only taking $\mu \to 0$ but also thinking of the blow-up coordinate of order $\mu^{\frac{1}{3}}$ near the Earth. This derivation, which is borrowed from [36], is given in section 3.3. The Hamiltonian of Hill's lunar problem is

$$H_{HLP} : T^*(\mathbb{R}^2 \setminus \{(0,0)\}) \to \mathbb{R},$$
$$H_{HLP}(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} + p_1q_2 - p_2q_1 - q_1^2 + \frac{1}{2}q_2^2.$$

Hill's lunar problem was introduced by Hill in order to study the stability of the motion of the Moon in [27]. Hill assumed that the Sun is infinitely far away from the Earth and has infinite mass. This approach brought us a simple Hamiltonian with great improvement in accuracy. As one can see, the difference on the Hamiltonians of the rotating Kepler problem and Hill's lunar problem is only the degree 2 term $-q_1^2+\frac{1}{2}q_2^2$. However, in the dynamics, this difference gives dramatic changes. For example, Hill's lunar problem is not completely integrable, see section 3.3 for the reference. Due to the non-integrability of Hill's lunar problem, geometric approaches could be effective for Hill's lunar problem. One common geometric feature of the rotating Kepler problem and Hill's lunar problem is fiberwise convexity.

Theorem for the fiberwise convexities of the rotating Kepler problem and Hill's lunar problem ([15] for the rotating Kepler problem, [32] for Hill's lunar problem)**.** Below the critical energy levels, the energy hypersurfaces of the rotating Kepler problem and Hill's lunar problem can be symplectically embedded into the cotangent bundle of S^2 as fiberwise convex hypersurfaces, respectively.

We will explain the meaning of fiberwise convexity in section 4.2. As soon as we know the fiberwise convexity of Hill's lunar problem, we can regard the regularized Hill's lunar problem as geodesic problems on S^2 with a family of Finsler metrics. Thus we can apply the results of Finsler geometry on S^2. For example, we can reprove the existence of two periodic orbits using the general result of Bangert and Long in [11]. We can obtain a systole bound using the universal systole bound on S^2 with Finsler metric proved in [9]. However, this universal systole bound, $\sqrt{3}\pi 10^8 \sqrt{Vol(\Sigma_H^c)}$, is not sharp enough. To improve the systole bound, one could try to get a pinching condition on the flag curvature in order to apply the result of Rademacher in [45]. For Hill's lunar problem, it is hard to compute the flag curvature of corresponding Finsler S^2 because of computational complexity. For a systole bound of Hill's lunar problem, we will sandwich Hill's lunar problem by the rotating Kepler problem and we will apply spectral invariants in the homology $H_*(\Lambda S^2)$ of the free loop space of S^2. As a result, we will prove the following Theorem.

Main Theorem. For the regularized Hill's lunar problem, there exists a (doubly symmetric) periodic orbit whose action is less than π for every energy value below the critical energy level.

This is a simple consequence of Theorem D1, D2 and D3 below. In order to apply spectral invariants of loop space homology, it is enough to consider the Morse homology on the free loop space ΛS^2 of S^2 in this restricted case because we have Finsler metrics for both of the rotating Kepler problem and Hill's lunar problem and so we can associate the energy functional

$$E(\gamma) = \frac{1}{2}\int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt$$

on ΛS^2. However, if we use symplectic homology instead of using Morse homology, this can be applied not only to fiberwise convex energy hypersurfaces but also to fiberwise star-shaped energy hypersurfaces. For example, the restricted three body problem has fiberwise star-shaped energy hypersurfaces but it is not known if it has fiberwise convex energy hypersurfaces. Therefore, we will use spectral invariants in the symplectic homology of fiberwise convex domains in a cotangent bundle. Moreover, we will consider spectral invariants in the wrapped Floer homology of fiberwise convex domains in a cotangent bundle in order to equip the symmetric notion.

Hamiltonian spectral invariants for Floer homology have been established by Schwarz for the symplectically aspherical case in [50] and Oh for the general case in [42]. Oh in [41] and Milinković in [38] defined a Lagrangian version of spectral invariants for cotangent bundles. Following the definitions in these papers, we will define spectral invariants in symplectic homology(section 6.1) and wrapped Floer homology(section 6.2) of fiberwise star-shaped domains in a cotangent bundle. However, the main different point in this thesis is that we assign the spectral invariant for Liouville domains in a fixed cotangent bundle. Let (N, g) be a closed orientable Riemannian manifold. Denote by $FSD(N)$ the *set of all fiberwise star-shaped domain in* T^*N. Since $M \in FSD(N)$ is a Liouville domain, we can define the symplectic homology of M. Moreover, if we consider submanifolds Q_0, Q_1 of N, then we can define the wrapped Floer homology of M with respect to the conormal bundles ν^*Q_0, ν^*Q_1(Definition 2.3.3). We have the long exact sequences

$$\cdots \to SH_*^{<b}(M) \xrightarrow{i_M^b} SH_*(M) \xrightarrow{j_M^b} SH_*^{\geq b}(M) \to SH_{*-1}^{<b}(M) \xrightarrow{i_M^b} \cdots$$

and

$$\cdots \to WFH_*^{<b}(\nu^*Q_0, \nu^*Q_1) \xrightarrow{i_M^b} WFH_*(\nu^*Q_0, \nu^*Q_1) \xrightarrow{j_M^b} WFH_*^{\geq b}(\nu^*Q_0, \nu^*Q_1)$$

$$\to WFH_{*-1}^{<b}(\nu^*Q_0, \nu^*Q_1) \xrightarrow{i_M^b} \cdots$$

for the symplectic homology and the wrapped Floer homology of M with action filtration b. Moreover, we have the isomorphism

$$\Psi_M : H_*(\Lambda N) \to SH_*(M)$$

between the homology of the free loop space of N and the symplectic homology of M, see [1], [49] and [52]. Similarly we have the isomorphism

$$\Psi_M : H_*(\mathcal{P}_{Q_0,Q_1}N) \to WFH_*(\nu^*Q_0, \nu^*Q_1, M)$$

between the homology of the space of paths from Q_0 to Q_1 on N and the wrapped Floer homology of M with respect to ν^*Q_0, ν^*Q_1, see [2]. With these ingredients, we can define maps

$$c_N : FSD(N) \times H_*(\Lambda N)^{\times} \to \mathbb{R},$$
$$c_N(M, \alpha) := \inf\{b \in \mathbb{R} \cup \{+\infty\} | \Psi_M(\alpha) \in \mathrm{im}(i^b_M)\}$$

and

$$c_{Q_0,Q_1,N} : FSD(N) \times H_*(\mathcal{P}_{Q_0,Q_1}N)^{\times} \to \mathbb{R},$$
$$c_{Q_0,Q_1,N}(M, \alpha) := \inf\{b \in \mathbb{R} \cup \{+\infty\} | \Psi_M(\alpha) \in \mathrm{im}(i^b_M)\}.$$

We will prove Theorem A1 and A2 in section 6.1 and 6.2.

Theorem A1 (Properties of c_N)**.** The map

$$c_N : FSD(N) \times H_*(\Lambda N)^{\times} \to \mathbb{R}, \quad (M, \alpha) \mapsto c(M, \alpha)$$

satisfies the following properties.

1) (Conformality) $c_N(kM, \alpha) = kc_N(M, \alpha)$ for all $k \in \mathbb{R}^+$.
2) (Monotonicity) $c_N(M_2, \alpha) \geq \kappa_{min}(\Sigma_1, \Sigma_2)c_N(M_1, \alpha)$ for all $M_1, M_2 \in FSD(N)$ where $\Sigma_i = \partial M_i$, $i = 1, 2$ and $\kappa_{min}(\Sigma_1, \Sigma_2) = \min_{x\in\Sigma_1}\{\kappa(x) | \kappa(x)x \in \Sigma_2, \kappa(x) > 0\}$.
3) (Spectrality) $c_N(M, \alpha) \in Spec(\Sigma, \lambda_{can})$ where $\Sigma = \partial M$.

for each $\alpha \in H_*(\Lambda N)^{\times}$.

Theorem A2 (Properties of $c_{Q_0,Q_1,N}$)**.** The map

$$c_{Q_0,Q_1,N} : FSD(N) \times H_*(\mathcal{P}_{Q_0,Q_1}N)^{\times} \to \mathbb{R}, \quad (M, \alpha) \mapsto c_{Q_0,Q_1,N}(M, \alpha)$$

satisfies the following properties.

1) (Conformality) $c_{Q_0,Q_1,N}(kM,\alpha) = kc_{Q_0,Q_1,N}(M,\alpha)$ for all $k \in \mathbb{R}^+$.

2) (Monotonicity) $c_{Q_0,Q_1,N}(M_2,\alpha) \geq \kappa_{min}(\Sigma_1,\Sigma_2)c_{Q_0,Q_1,N}(M_1,\alpha)$ for all $M_1, M_2 \in FSD(N)$ where $\Sigma_i = \partial M_i$, $i = 1,2$ and $\kappa_{min}(\Sigma_1,\Sigma_2) = \min_{x\in\Sigma_1}\{\kappa(x)|\kappa(x)x \in \Sigma_2, \kappa(x) > 0\}$.

3) (Spectrality) $c_{Q_0,Q_1,N}(M,\alpha) \in Spec(\Sigma, \lambda_{can}; \partial\mathcal{L}_0, \partial\mathcal{L}_1)$ where $\Sigma = \partial M$ and $\mathcal{L}_i = M \cap \nu^* Q_i$.

for each $\alpha \in H_*(\mathcal{P}_{Q_0,Q_1}N)^\times$.

We denote by Σ_R^c and $\Sigma_H^{c'}$ the regularized energy hypersurface of the rotating Kepler problem of energy $-c$ and Hill's lunar problem at energy $-c'$, respectively. Since they are fiberwise convex, they bound Liouville domains M_R^c and $M_H^{c'}$, respectively. We define increasing sequences

$$c_R^P := \frac{P+3}{2(P+1)^{\frac{1}{3}}}, \quad c_H^P := \frac{2P+8-\sqrt{(P+1)(P+9)}}{2(P+1)^{\frac{1}{3}}}$$

for $P = 1,2,3,\cdots$. We define by $-c_R^0 = -\frac{3}{2}$ and $-c_H^0 = -\frac{3^{\frac{4}{3}}}{2}$ the critical values of the rotating Kepler problem and Hill's lunar problem. We will prove the following Theorem.

Theorem B. For the fiberwise convex domains M_R^c and $M_H^{c'}$ in T^*S^2 defined by the regularized energy hypersurfaces of the rotating Kepler problem and Hill's lunar problem, we have the following inclusions in T^*S^2.
(1) $M_H^c \subset M_R^{c_R^1}$ for all $c \geq c_H^0$.
(2) $M_H^c \subset M_R^{c_R^P}$ if $c \geq c_H^P$ for all $P = 2,3,4,\cdots$.
(3) $M_R^{c+\frac{1}{2c^2}} \subset M_H^c$ for all $c > c_H^0$.

In section 7.1, we recall Conley-Zehnder indices of all periodic orbits of the rotating Kepler problem from [7]. We also compute the actions of all periodic orbits of the rotating Kepler problem in section 7.2. Based on these computations, we can compute the spectral invariant $c_{S^2}(M_R^c, \Delta)$ for some $\Delta \in H_*(\Lambda S^2)$.

Theorem C1. Let P be a positive integer. If $c > c_R^P$, then there exist homology classes $\Delta_{R,N} \in H_*(\Lambda S^2)$ for $N = 1, 2, \cdots, P+1$ and $\Delta_{D,N} \in H_*(\Lambda S^2)$ for $N = 1, 2, \cdots, P$ such that

$$c_{S^2}(M_R^c, \Delta_{R,N}) = 2\pi L_R(c) N, \quad c_{S^2}(M_R^c, \Delta_{D,N}) = -2\pi L_D(c) N$$

where the values

$$0 < L_R(c) = \frac{1}{2}\sqrt{\frac{3}{2c}} \sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2c}\right)^{\frac{3}{2}}\right)\right),$$

$$-1 < L_D(c) = \frac{1}{2}\sqrt{\frac{3}{2c}} \sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2c}\right)^{\frac{3}{2}}\right) + \frac{2\pi}{3}\right) < 0$$

are zero of $c = \frac{1}{2x^2} - x$.

For the corresponding result for the wrapped Floer homology, we define equators Q_1, Q_2 of S^2. If we consider the conormal bundles $\nu^* Q_1, \nu^* Q_2$, then they are the fixed point sets of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. We will define anti-symplectic involutions $\mathcal{R}_1$ and $\mathcal{R}_2$ in section 3.3. Moreover, we can easily see that the rotating Kepler problem and Hill's lunar problem are invariant under $\mathcal{R}_1$ and $\mathcal{R}_2$. In section 7.1, we compute Robbin-Salamon indices of all Hamiltonian chords from $\nu^* Q_1$ to itself or to $\nu^* Q_2$ of the rotating Kepler problem. Likewise, we can compute the spectral invariant $c_{Q_1,S^2}(M_R^c, \Xi)$ for some $\Xi \in H_*(\mathcal{P}_{Q_1} S^2)$ and the spectral invariant $c_{Q_1,Q_2,S^2}(M_R^c, \Pi)$ for some $\Pi \in H_*(\mathcal{P}_{Q_1,Q_2} S^2)$.

Theorem C2. Let P be a positive integer. If $c > c_R^P$, then there exist homology classes $\Xi_{R,N} \in H_*(\mathcal{P}_{Q_1} S^2)$ for $N = 1, 2, \cdots, P+1$ and $\Xi_{D,N} \in H_*(\mathcal{P}_{Q_1} S^2)$ for $N = 1, 2, \cdots, P$ such that

$$c_{Q_1,S^2}(M_R^c, \Xi_{R,N}) = \pi L_R(c) N, \quad c_{Q_1,S^2}(M_R^c, \Xi_{D,N}) = -\pi L_D(c) N.$$

Theorem C3. If $c > c_R^1$, then there exist $\Pi_R \in H_*(\mathcal{P}_{Q_1,Q_2} S^2)$ and $\Pi_D \in$

$H_*(\mathcal{P}_{Q_1,Q_2}S^2)$ such that

$$c_{Q_1,Q_2,S^2}(M_R^c,\Pi_R)=\frac{1}{2}\pi L_R(c)N,\quad c_{Q_1,Q_2,S^2}(M_R^c,\Pi_D)=-\frac{1}{2}\pi L_D(c)N.$$

The proofs of Theorem C1, C2 and C3 are given in section 7.3.

Organizing Theorem A1, A2, B, C1 C2 and C3, we can obtain estimates of spectral invariants of M_H^c in T^*S^2. Finally, we state the goal of this thesis.

Theorem D1. We have the following estimates for $c_{S^2}(M_H^c,\cdot)$
(1) The inequalities

$$2\pi\frac{-1+\sqrt{1+8c^3}}{4c^2}\leq c_{S^2}(M_H^c,\Delta_{R,1})<2\pi\times 0.490534,$$
$$2\pi\frac{1+\sqrt{1+8c^3}}{4c^2}\leq c_{S^2}(M_H^c,\Delta_{D,1})<2\pi\times 0.793701$$

hold for all $c>c_H^0$.
(2) If $c\in[c_H^P,c_H^{P+1})$ for some $P\in\{2,3,4,\cdots\}$, then the inequalities

$$2\pi N\frac{-1+\sqrt{1+8c^3}}{4c^2}\leq c_{S^2}(M_H^c,\Delta_{R,N})\leq 2\pi N\frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}},$$
$$2\pi N\frac{1+\sqrt{1+8c^3}}{4c^2}\leq c_{S^2}(M_H^c,\Delta_{D,N})\leq 2\pi N(P+1)^{-\frac{1}{3}}$$

hold for all $N=1,2,\cdots,P$.

Theorem D2. We have the following estimates for $c_{Q1,S^2}(M_H^c,\cdot)$
(1) The inequalities

$$\pi\frac{-1+\sqrt{1+8c^3}}{4c^2}\leq c_{Q_1,S^2}(M_H^c,\Xi_{R,1})<\pi\times 0.490534,$$
$$\pi\frac{1+\sqrt{1+8c^3}}{4c^2}\leq c_{Q_1,S^2}(M_H^c,\Xi_{D,1})<\pi\times 0.793701$$

hold for all $c>c_H^0$.

(2) If $c \in [c_H^P, c_H^{P+1})$ for some $P \in \{2, 3, 4, \cdots\}$, then the inequalities

$$\pi N \frac{-1+\sqrt{1+8c^3}}{4c^2} \leq c_{Q_1,S^2}(M_H^c, \Xi_{R,N}) \leq \pi N \frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}},$$

$$\pi N \frac{1+\sqrt{1+8c^3}}{4c^2} \leq c_{Q_1,S^2}(M_H^c, \Xi_{D,N}) \leq \pi N (P+1)^{-\frac{1}{3}}$$

hold for all $N = 1, 2, \cdots, P$.

Theorem D3. The inequalities

$$\frac{1}{2}\pi \frac{-1+\sqrt{1+8c^3}}{4c^2} \leq c_{Q_1,Q_2,S^2}(M_H^c, \Pi_R) < \frac{1}{2}\pi \times 0.490534,$$
$$\frac{1}{2}\pi \frac{1+\sqrt{1+8c^3}}{4c^2} \leq c_{Q_1,Q_2,S^2}(M_H^c, \Pi_D) < \frac{1}{2}\pi \times 0.793701$$

hold for all $c > c_H^0$.

Theorem D1 gives estimates for action values of periodic orbits of Hill's lunar problem. Theorem D2 and D3 give estimates for action values of symmetric periodic orbits and doubly symmetric orbits of Hill's lunar problem, respectively. Throughout this thesis, we will prove Theorem A1, A2, B, C1, C2, C3, D1, D2 and D3.

Chapter 2

Preliminaries

In this chapter, we will introduce basic notations, conventions and definitions which will be used throughout the thesis. We will also recall auxiliary properties of symplectic geometry and Hamiltonian dynamics.

2.1 Hamiltonian dynamics and Symplectic geometry

We begin with the simplest case of *configuration space*: all possible states of position. Consider a particle P of mass 1 in $\mathbb{R}^n$ under some force F. If we denote by $q \in \mathbb{R}^n$ the position of the particle, then the equation of motion is

$$\frac{d^2q}{dt^2} = F$$

by Newton's law. Assume that the force F depends on the position of particle and time. Additionally, assume that F has the *potential U, that is,* $-\nabla U(t,q) = F(t,q)$ where ∇ is the gradient for the variable q. Then the equation of motion becomes the second order differential equation

$$\frac{d^2q}{dt^2} = -\nabla U(t,q)$$

for $q \in \mathbb{R}^n$. We denote by $\dot{} = \frac{d}{dt}$ from now on. If we introduce the variable $v = \dot{q}$, then the equation becomes the first order differential equation

$$\begin{cases} \dot{q} = v, \\ \dot{v} = -\nabla U(t,q) \end{cases}$$

for $(q, v) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. This can be interpreted by principle of least action. We define the function

$$L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad L(t,q,v) = \frac{1}{2}|v|^2 - U(t,q)$$

which is called *Lagrangian.* Then it is easy to see that the above first order equation can be written by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(t,q,\dot{q})\right) = \frac{\partial L}{\partial q}(t,q,\dot{q}) \tag{2.1.1}$$

and this equation is called the *Euler-Lagrange equation associated to L*. The Euler-Lagrange equation can be derived by calculus of variations. Fix the positions $q_0, q_1 \in \mathbb{R}^n$ and the times $t_0, t_1 \in \mathbb{R}$. We define the following functional

$$\mathcal{E} : \mathcal{P}_{q_0,q_1}(\mathbb{R}^n) \to \mathbb{R}, \quad \mathcal{E}(\gamma) = \int_{t_0}^{t_1} L(t,\gamma,\dot{\gamma})dt \tag{2.1.2}$$

on the path space $\mathcal{P}_{q_0,q_1}(\mathbb{R}^n) := \{\gamma \in C^\infty([t_0,t_1],\mathbb{R}^n) | \gamma(t_0) = q_0, \gamma(t_1) = q_1\}$ from q_0 to q_1 for $\gamma \in \mathcal{P}_{q_0,q_1}(\mathbb{R}^n)$. This functional is called the *energy functional associated to L*. One can show that solutions of the Euler-Lagrangian (2.1.1) are stationary points of the energy functional (2.1.2), see [43] for proof.

Definition 2.1.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *superlinear*, if f is bounded below and satisfies

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|} = +\infty.$$

Definition 2.1.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a superlinear function. The *Legendre* *transformation of* f is the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(y) := \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - f(x)).$$

We denote by $g = \mathcal{L}(f)$ the Legendre transformation of f. Here, $\langle \cdot, \cdot \rangle$ is the standard inner product.

We use the fact that the Legendre transformation is well-defined provided f is superlinear. Moreover, we will consider only simple case, that is, f is differentiable and convex. In this case, the Legendre transformation is well-defined and satisfies the following property. One can find more details in the book [43].

Theorem 2.1.3 (Fenchel). *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a differentiable, superlinear and convex function. Then the Legendre transformation* $g = \mathcal{L}(f)$ *is convex and* $f = \mathcal{L}(g)$.

Proof. For $y_1, y_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} g(\lambda y_1 + (1-\lambda) y_2) &= \sup_{x \in \mathbb{R}^n} (\langle \lambda y_1 + (1-\lambda) y_2, x \rangle - f(x)) \\ &= \sup_{x \in \mathbb{R}^n} (\lambda(\langle y_1, x \rangle - f(x)) + (1-\lambda)(\langle y_2, x \rangle - f(x))) \\ &\leq \sup_{x \in \mathbb{R}^n} (\lambda(\langle y_1, x \rangle - f(x)) + \sup_{x \in \mathbb{R}^n} (1-\lambda)(\langle y_2, x \rangle - f(x))) \\ &= \lambda g(y_1) + (1-\lambda) g(y_2) \end{aligned}$$

and so g is convex. If f is convex, then the function $x \mapsto \langle x, y \rangle - f(x)$ is a concave function and goes to $-\infty$ as $|x|$ goes $+\infty$ for each $y \in \mathbb{R}^n$. Hence we have a unique maximum and it is attained at x if and only if $\nabla f(x) = y$. Thus, we have

$$g(y) \geq \langle x, y \rangle - f(x)$$

and equality holds if and only if $y = \nabla f(x)$. This implies that $f(x) \geq \mathcal{L}(g)(x)$. On the other hand, $f(x) = \langle \bar{y}, x \rangle - g(\bar{y})$ for $\bar{y} = \nabla f(x)$ implies $f(x) \leq \mathcal{L}(g)(x)$. This proves Theorem 2.1.3 $\square$

Now we assume that the Lagrangian function L is smooth and fiberwise convex, namely, the map $v \mapsto L(t, q, v)$ is convex for each fixed (t, q). We can apply the Legendre transformation

$$H(t,q,p) := \mathcal{L}(L(t,q,v)) = \sup_{v\in\mathbb{R}^n}(\langle p,v\rangle - L(t,q,v))$$

for v-variable. Then we have the following Theorem.

Theorem 2.1.4. *Let L be a smooth and fiberwise convex Lagrangian. The curve $q \in C^\infty([t_0,t_1],\mathbb{R}^n)$ satisfies the Euler-Lagrange equation*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(t,q,\dot{q})\right) = \frac{\partial L}{\partial q}(t,q,\dot{q})$$

if and only if the curve $(q,p) \in C^\infty([t_0,t_1],\mathbb{R}^n\times\mathbb{R}^n)$ satisfies the Hamiltonian equation

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p}(t,q,p), \\ \dot{p} = -\frac{\partial H}{\partial q}(t,q,p) \end{cases}$$

where $H = \mathcal{L}(L)$ and $p = \frac{\partial L}{\partial v}(t,q,\dot{q})$.

Proof. From the computation in Theorem 2.1.3, we have $H(t,q,p) = \langle p,v\rangle - L(t,q,v)$ for $p = \frac{\partial L}{\partial v}(t,q,v)$. Since L is the Legendre transformation of H by Theorem 2.1.3, we have that $v = \frac{\partial H}{\partial p}$. Thus the equation $\dot{q} = v = \frac{\partial H}{\partial p}$ holds along the solution curve. From the identity

$$L(t,q,v) = \sup_{p\in\mathbb{R}^n}(\langle v,p\rangle - H(t,q,p)) = \left\langle v, \frac{\partial L}{\partial v}\right\rangle - H(t,q,\frac{\partial L}{\partial v}),$$

we have the following identity

$$\frac{\partial L}{\partial q} = v\frac{\partial^2 L}{\partial q\partial v} - \frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial^2 L}{\partial q\partial v} = -\frac{\partial H}{\partial q}.$$

This implies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial q} \iff \dot{p} = -\frac{\partial H}{\partial q}$$

for the corresponding p, v. This proves Theorem 2.1.4 □

We call the variable $p = \frac{\partial L}{\partial v}$ the *generalized momentum*. The function H is called *Hamiltonian*.

Example 2.1.5. A Lagrangian $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a *mechanical Lagrangian*, if it has the following form

$$L(t, q, v) = \frac{1}{2} m|v|^2 - U(t, q)$$

where m is the mass of particle and so a constant. Then the generalized momentum is $p = \frac{\partial L}{\partial v} = mv$. The Legendre transformation $H = \mathcal{L}(L)$ is given by the total energy

$$H(t, q, p) = \frac{|p|^2}{2m} + U(t, q).$$

This form of Hamiltonian is called a *mechanical Hamiltonian*.

Consider a Hamiltonian $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the associated Hamiltonian equation

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p}(t, q, p), \\ \dot{p} = -\frac{\partial H}{\partial q}(t, q, p). \end{cases}$$

We can write it as a vector notation $\dot{x} = X_H(t, x)$ by defining $x = (q, p)$ and $X_H(t, x) = (\frac{\partial H}{\partial p}(t, x), -\frac{\partial H}{\partial q}(t, x))$. The vector field X_H is called the *Hamiltonian vector field of H*. If we think of the 2-form $\omega_0 = dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$ on $\mathbb{R}^n \times \mathbb{R}^n$, then the vector field X_H^t can be obtained by

$$\iota_{X_H^t} \omega_0 = -dH_t$$

where $H_t(x) := H(t, x)$ is the function on the phase space $\mathbb{R}^n \times \mathbb{R}^n$. This 2-form ω_0 plays the role of switching vectors and covectors unique way due to its nondegeneracy. This exposition allows us to extend the phase space into arbitrary symplectic manifolds.

For an intermediate step, we consider the cotangent bundle T^*N of a manifold N. In fact, the cotangent bundle case is the main object in this thesis. We will show that there is always the canonical way to convert covectors with vectors on any cotangent bundle. In other word, every cotangent bundle has canonical 1-form and its differential can play the same role with ω_0. We define this canonical 1-form.

Definition 2.1.6. Let $\pi : T^*N \to N$ be the canonical projection. The *Liouville 1-form(or canonical 1-form)* λ_{can} is defined by

$$\lambda_{can}(v) = p(\pi_*(v))$$

for $v \in T_xT^*N$ where $x = (q,p) \in T^*N$ with $q = \pi(x) \in M$ and $p \in T^*_qN$.

In canonical coordinates (q,p), that is, q-variables are coordinates on N and p-variables are the conjugated momentum, we can express these forms

$$\lambda_{can} = pdq, \quad \omega_{can} = dp \wedge dq$$

in terms of q, p. It is independent of the choice of canonical coordinates. This implies that in any local coordinate system of N, the equivalence between the Euler-Lagrange equation and the Hamiltonian equation can be derived by exactly same manner with $\mathbb{R}^n$ case. Thus, we can define the Hamiltonian equation with coordinate free notation. Let $H : \mathbb{R} \times T^*N \to \mathbb{R}$ be a Hamiltonian. We define the *Hamiltonian vector field X^t_H associated to H* by

$$\iota_{X^t_H}\omega_{can} = -dH_t$$

where $H_t(x) = H(t,x)$. The first order differential equation

$$\dot{x}(t) = X^t_H(x(t)), \quad \text{for} \quad x(t) \in T^*N$$

is called the *Hamiltonian equation associated to H*. In canonical coordinates system (q,p), the Hamiltonian vector field has the form

$$X_H(t,q,p) = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}.$$

One can easily see that ω_{can} is closed and nondegenerate. The 2-form ω_{can}

is called the *canonical symplectic form*. This fact leads us naturally to the definition of symplectic manifolds.

Definition 2.1.7. A smooth manifold M equipped with a 2-form ω is called a *symplectic manifold* if ω is closed and nondegenerate. The 2-form ω is called a *symplectic form*.

Definition 2.1.8. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A smooth map $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is called *symplectic* if $\phi^*\omega_2 = \omega_1$. In addition, if ϕ is a diffeomorphism, then ϕ is called a *symplectomorphism*.

Even if we are mostly interested in the cotangent bundle case, it is not difficult to develop the properties of Hamiltonian equations on symplectic manifolds. Thus we will discuss about the Hamiltonian equation on symplectic manifolds. Suppose that (M, ω) is a symplectic manifold. A Hamiltonian is a function on $\mathbb{R} \times M$. Let H be a Hamiltonian on M. We will write $H_t := H(t, \cdot) : M \to \mathbb{R}$ for notational convenience. Following the cotangent bundle case, we can define the *Hamiltonian vector field X_H^t associated to H* by

$$\iota_{X_H^t}\omega = -dH_t$$

and this is uniquely defined by nondegeneracy of ω.

The *Hamiltonian flow* ϕ_H^t is the flow of the Hamiltonian vector field and so defined by the differential equation

$$\frac{d}{dt}\phi_H^t(x) = X_H^t(\phi_H^t(x)), \phi_H^0(x_0) = x_0$$

and $\phi_H^t(x_0)$ is given by solving the initial value problem

$$\dot{x}(t) = X_H^t(x(t)), \quad x(0) = x_0 \in M.$$

This equation is called the *Hamiltonian equation associated to H*. We call this diffeomorphism ϕ_H^t a *Hamiltonian diffeomorphism generated by H at time t* for each fixed t. Hamiltonian diffeomorphisms satisfy some properties. Let us check the following basic properties.

Theorem 2.1.9. *The Hamiltonian diffeomorphism ϕ_H^t is a symplectomorphism for each t.*

Proof. Since ϕ_H^0 is the identity, it is enough to see that $(\phi_H^t)^*\omega$ is time independent. Hence we will show that $\frac{d}{dt}(\phi_H^t)^*\omega = 0$ for any t. Using Cartan's formula, we have that

$$\frac{d}{dt}(\phi_H^t)^*\omega = (\phi_H^t)^* L_{X_H^t}\omega = (\phi_H^t)^*(d\iota_{X_H^t}\omega + \iota_{X_H^t}d\omega)$$

and $\iota_{X_H^t}d\omega = 0$, $d\iota_{X_H^t}\omega = d(-dH_t) = 0$ from the closedness of ω and the definition of Hamiltonian vector field. This implies that $\frac{d}{dt}(\phi_H^t)^*\omega = 0$. □

Let $Symp(M,\omega)$ be the *group of all symplectomorphisms on the symplectic manifold* (M,ω). We denote by

$$Ham(M,\omega) := \{\phi_H^1 | H \in C^\infty([0,1]\times M)\}$$

the *set of all Hamiltonian diffeomorphisms*. Then Theorem 2.1.9 tells us that $Ham(M,\omega) \subset Symp(M,\omega)$. It is clear that the set $Symp(M,\omega)$ forms a group with the composition of two maps. But it is not obvious if $Ham(M,\omega)$ is a subgroup of $Symp(M,\omega)$. The following Lemma will prove that $Ham(M,\omega)$ is a subgroup of $Symp(M,\omega)$.

Lemma 2.1.10. *For given Hamiltonians $K, H \in C^\infty(\mathbb{R}\times M)$, we define a Hamiltonian $K\#H \in C^\infty(\mathbb{R}\times M)$ by $K\#H(t,x) = K(t,x) + H(t,(\phi_K^t)^{-1}(x))$. Then we have*

$$\phi_{K\#H}^t = \phi_K^t \circ \phi_H^t$$

for all $t \in \mathbb{R}$.

Proof. Let x be a point in M and $y = \phi_K^t \circ \phi_H^t(x)$. We take any tangent vector $v \in T_yM$ at y. Then we have

$$\begin{aligned}
\omega(\frac{d}{dt}\phi_K^t \circ \phi_H^t(x), v) &= \omega(X_K^t(y) + d\phi_K^t(\phi_H^t(x))(X_H^t(x)), v) \\
&= \omega(X_K^t(y), v) + \omega(d\phi_K^t(\phi_H^t(x))(X_H^t(x)), v) \\
&= -dK_t(v) + \omega(X_H^t(x), (d\phi_K^t(\phi_H^t(x))^{-1}(v)) \\
&= -dK_t(v) - dH_t((d\phi_K^t(\phi_H^t(x))^{-1}(v)) \\
&= -(dK_t + d(H \circ (\phi_K^t)^{-1}))(v) \\
&= -d(K\#H)(v)
\end{aligned}$$

using the fact that ϕ_K^t is a symplectomorphism at the third equality. This implies that

$$\frac{d}{dt}\phi_K^t \circ \phi_H^t(x) = \frac{d}{dt}\phi_{K\#H}^t(x)$$

and so $\phi_{K\#H}^t = \phi_K^t \circ \phi_H^t$ because $\phi_K^0 \circ \phi_H^0 = Id = \phi_{K\#H}^0$. This proves Lemma 2.1.10. $\square$

Remark 2.1.11. Let K, H be Hamiltonians on (M, ω). Since the composition of two Hamiltonian diffeomorphisms $\phi_K^1, \phi_H^1 \in Ham(M, \omega)$ is again a Hamiltonian diffeomorphism $\phi_{K\#H}^1 \in Ham(M, \omega)$, the subset $Ham(M, \omega)$ is closed under the group operation of $Symp(M, \omega)$. Moreover, if we use Lemma 2.1.10 again, it is not hard to see that $\bar{K}(t, x) = -K(t, \phi_K^t(x))$ generates the Hamiltonian diffeomorphism which is inverse of ϕ_K^t for any $K \in C^\infty([0, 1] \times M)$. This implies that $Ham(M, \omega)$ is a subgroup of $Symp(M, \omega)$.

We will mostly deal with time-independent Hamiltonian in this thesis. When a Hamiltonian H is time-independent, the energy is conserved due to the time translation symmetry. This can be followed by simple computation below.

Theorem 2.1.12 (Energy Conservation)**.** *If H is time-independent, that is, H is a function on M, then the Hamiltonian flow ϕ_H^t preserves energy, that is, $H(\phi_H^t(x)) = H(x)$ for all $t \in \mathbb{R}$ and $x \in M$.*

Proof. The proof can be done by the following equation

$$\begin{aligned}\frac{d}{dt}H(\phi_H^t(x)) &= dH(\phi_H^t(x))[\frac{d}{dt}\phi_H^t(x)] = dH(\phi_H^t(x))[X_H(\phi_H^t(x))]\\ &= -\omega(\phi_H^t(x))(X_H(\phi_H^t(x)), X_H(\phi_H^t(x))) = 0\end{aligned}$$

for any $t \in \mathbb{R}$ and $x \in M$. This proves Theorem 2.1.12. □

Theorem 2.1.12 tells us that if H is a time-independent Hamiltonian and $x \in H^{-1}(c)$, then $\phi_H^t(x) \in H^{-1}(c)$ for every t. In other words, the Hamiltonian vector field X_H is tangential to the energy hypersurface of H. Thus the energy hypersurface $\Sigma := H^{-1}(c)$ is foliated by leaves $L_x := \{\phi_H^t(x) | t \in \mathbb{R}\}$ through $x \in \Sigma$. At first glance, it seems that this foliation depends on the choice of Hamiltonian. Surprisingly the foliation is completely determined by the hypersurface Σ itself. We prove this as follows. Suppose that Σ is a hypersurface, codimension 1 submanifold, of a symplectic manifold (M, ω). Then Σ induces the *canonical line bundle over* Σ

$$L_\Sigma \to \Sigma$$

as a subbundle of the tangent bundle TM by defining the fiber

$$L_{\Sigma,x} = \{v \in T_xM | \omega(v, w) = 0 \text{ for all } w \in T_x\Sigma\}$$

as the symplectic complement of $T_x\Sigma$ for each $x \in \Sigma$. Since $\dim T_x\Sigma + \dim T_x\Sigma^\omega = \dim T_xM$, L_Σ becomes a line bundle. Moreover, every hyperplane is a coisotropic subspace in the symplectic space, that is, $L_{\Sigma,x} = T_x\Sigma^\omega \subset T_x\Sigma$ for each $x \in \Sigma$. Therefore, the line bundle L_Σ is a line subbundle of $T\Sigma$. We will see that the Hamiltonian vector field of a time-independent Hamiltonian is a section of the canonical line bundle on each energy hypersurface.

Lemma 2.1.13. *Let $H : M \to \mathbb{R}$ be a time-independent Hamiltonian on a symplectic manifold (M, ω). Then the Hamiltonian vector field X_H on a energy hypersurface $\Sigma = H^{-1}(c)$ defines a section of the canonical line bundle $L_\Sigma \to \Sigma$.*

Proof. We have to show that $X_H(x) \in L_{H^{-1}(c),x}$ for each $x \in H^{-1}(c)$. By definition, it is enough to see that $\omega(X_H(x), w) = 0$ for all $w \in T_x H^{-1}(c)$. In fact, for any $w \in T_x H^{-1}(c)$ we have

$$\omega(X_H(x), w) = -dH(x)[w] = 0.$$

This proves Lemma 2.1.13. $\square$

Lemma 2.1.13 implies that if two Hamiltonians have same regular energy hypersurface than the Hamiltonian flows are same on that energy hypersurface up to reparametrization. In other words, they are orbitally equivalent. For example, if we composite a monotone increasing invertible function $f : \mathbb{R} \to \mathbb{R}$ to a given Hamiltonian H, then $X_{f \circ H}$ is parallel to X_H and so has the same Hamiltonian flow up to reparametrization.

2.2 Integrals and Completely Integrable Systems

We showed the law of energy conservation for time-independent Hamiltonians through Theorem 2.1.12. Sometimes Hamiltonian system could have some conservative quantities. For a given Hamiltonian $H : \mathbb{R} \times M \to \mathbb{R}$, if a smooth function $F : M \to \mathbb{R}$ is preserved under the Hamiltonian flow $\phi_H^t(x)$, namely $F(x) = F(\phi_H^t(x))$ for all $x \in M$ and $t \in \mathbb{R}$, then we call F an *integral of the Hamiltonian system* X_H. In order to discuss integrals of Hamiltonian systems, it is convenient to introduce the notion of Poisson bracket.

Definition 2.2.1. Let F and G be smooth functions on (M, ω). We define the *Poisson bracket of F and G* by

$$\{F, G\} = \omega(X_F, X_G) = -dF(X_G) = dG(X_F) = -X_G(F) = X_F(G)$$

We can immediately see the following Lemma from the definition.

Lemma 2.2.2. *The function $F \in C^\infty(M)$ is an integral of X_H if and only if $\{H, F\} = 0$.*

Proof.

$$\begin{aligned} & F(x) = F(\phi_H^t(x)) \quad \text{for all} \quad t \in \mathbb{R} \\ \iff & \frac{d}{dt}F(\phi_H^t(x)) = 0 \iff dF(X_H) = 0 \iff \{H, F\} = 0 \end{aligned}$$

□

We say that F and G are Poisson commute, if $\{F, G\} = 0$. Lemma 2.2.2 says that an integral of X_H is a Poisson commuting function with H. By the skew-symmetric property of the Poisson bracket, F is an integral of X_H if and only if H is an integral of X_F. We say that a flow ϕ^t is a *symmetry for* H if $H(x) = H(\phi^t(x))$ for all $x \in M$ and $t \in \mathbb{R}$. The following interesting phenomenon was discovered by Noether.

Theorem 2.2.3 (Noether)**.** *Let H be a time-independent Hamiltonian on a symplectic manifold (M, ω). Then $F \in C^\infty(M)$ is an integral of X_H if and only if the flow ϕ_F^t is a symmetry for H.*

Therefore, whenever we have a symmetry, we can obtain an integral. For example, the rotational symmetry gives the integral of angular momentum and the translation symmetry gives the integral of momentum.

Example 2.2.4 (The planar Kepler problem)**.** We consider the Hamiltonian $H_{KP}(q, p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$ on symplectic manifold $((\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}^2, \omega = dp \wedge dq)$ where $q \in \mathbb{R}^2 - \{0\}$ denotes the position variable and $p \in \mathbb{R}^2$ denotes the momentum variable. This Hamiltonian system defines the Kepler problem on a plane. Obviously, this problem has the rotational symmetry, that is, this Hamiltonian H_{KP} is invariant under the flow

$$\Psi^t := R^t \oplus R^t$$

where $R^t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ is the flow applied on $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ variables, respectively. We claim that the function, the angular momentum,

$$L := q_1 p_2 - q_2 p_1$$

generates the flow Ψ^t, that is, $\phi_L^t(q,p) = \Psi^t(q,p)$ for all $(q,p) \in (\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}^2$ and $t \in \mathbb{R}$. From the differential

$$dL = p_2 dq_1 - p_1 dq_2 - q_2 dp_1 + q_1 dp_2$$

of L, we obtain the Hamiltonian vector field

$$X_L(q,p) = -q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}$$

of L and this generates the rotation transformation Ψ^t. Therefore, the function L is an integral of the planar Kepler problem of the Hamiltonian H_{KP}. Alternatively, we can check that

$$\{H_{KP}, L\} = -dH_{KP}(X_L) = 0$$

in order to prove L is an integral of the planar Kepler problem.

An integral of a Hamiltonian system helps to solve the system. An integral reduces the dimension of the Hamiltonian system by 1 dimension. The extreme case having maximal number of integrals is called *completely integrable system.*

Definition 2.2.5. Let H be a Hamiltonian on the symplectic manifold (M,ω) of dimension $2n$. The Hamiltonian system with Hamiltonian H is called *completely integrable* if there exist smooth functions $F_1, F_2, \cdots, F_n \in C^\infty(M)$ such that

- $F_1, \cdots, F_n$ are integrals of X_H,
- $\{dF_1(x), \cdots, dF_n(x)\}$ is linearly independent on T_x^*M for almost every $x \in M$,
- The Poisson brackets commute pairwise $\{F_i, F_j\} = 0$ for all $i, j \in \{1, 2, \cdots, n\}$,
- The Hamiltonian vector fields X_{F_i} are complete for all $i \in \{1, 2, \cdots, n\}$.

The planar Kepler problem is an example of completely integrable system by setting $F_1 = H_{KP}$, $F_2 = L$. We will see the rotating Kepler problem is also completely integrable. Such completely integrable systems are relatively easy to understand their dynamics due to the following Theorem. One can find the proof of the following Theorem, for example in [10].

Theorem 2.2.6 (Arnold-Liouville)**.** *Let the Hamiltonian system of H on (M, ω) be a completely integrable system with the set of integrals $\{F_1, F_2, \cdots, F_n\}$. Let $F : M \to \mathbb{R}^n$ denote the vector valued function $F = (F_1, F_2, \cdots, F_n)$. If $c \in \mathbb{R}^n$ is a regular value of F, then the level set $T_c = F^{-1}(c)$ satisfies the following.*

1. *T_c is a smooth Lagrangian submanifold.*
2. *If T_c is compact and connected, then T_c is diffeomorphic to the n-dimensional torus. This torus is called the Liouville torus.*
3. *In the neighborhood $U = \mathbb{T}^n \times D^n$ of the Liouville torus T_c, there is a coordinate system $s = (s_1, \cdots, s_n), \phi = (\phi_1, \cdots, \phi_n)$, where s is for D^n and ϕ is for $\mathbb{T}^n$, satisfying the following*
   - $\omega = \Sigma ds_i \wedge d\phi_i$,
   - the variables s_i are functions of $F_1, \cdots, F_n$,
   - the Hamiltonian vector field X_H is given by
   $$X_H(s, \phi) = f(s)\frac{\partial}{\partial \phi}$$
   in this coordinate system for some function $f : D^n \to \mathbb{R}^n$ and so the flow is $\phi_H^t(s, \phi) = (s, \phi + f(s)t)$. We call (s, ϕ) by action-angle coordinates.

For the first statement of Arnold-Liouville Theorem, we recall the definition of Lagrangian submanifold.

Definition 2.2.7. Let (M, ω) be a $2n$-dimensional symplectic manifold. An n-dimensional submanifold L of M is called a *Lagrangian submanifold* if $i^*\omega = 0$ where the map $i : L \hookrightarrow M$ is the inclusion.

2.3 Geodesic Problems

Let (N, g) be a Riemannian n-manifold. In dynamical aspect, one of the most interesting objects is the geodesic. The *geodesic equation on* (N, g) is a second order differential equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad i = 1, 2, \cdots, n$$

for a curve $x : I \to N$ in a local coordinate U of N where $\Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{ij,m})$ are *Christoffel symbols*. Here we follow the Einstein summation convention. This differential equation is derived from the Euler-Lagrange equations of motion of the *energy functional*

$$\mathcal{E} : \mathcal{P}_{x_0,x_1}(N) \to \mathbb{R}, \quad \mathcal{E}(\gamma) = \int_0^1 \frac{1}{2}g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt$$

on the smooth path space $\mathcal{P}_{x_0,x_1}(N) = \{\gamma \in C^\infty([0, 1], N)|\gamma(0) = x_0, \gamma(1) = x_1\}$ *connecting two points* $x_0, x_1 \in N$. In other words, a geodesic connecting two points x_0, x_1 is the stationary point of the energy functional $\mathcal{E}$ defined above. One can regard the integrand $\frac{1}{2}g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))$ as a Lagrangian L defined on the tangent bundle TN by

$$L : TN \to \mathbb{R}, \quad L(q, v) = \frac{1}{2}g_q(v, v)$$

where $q \in N, v \in T_qN$. Using the Legendre transformation, one can derive the Hamiltonian equation on the cotangent bundle T^*N corresponding to the Euler-Lagrangian equation and consequently to the geodesic equation. Definition 2.1.6 tells us that every cotangent bundle has the canonical symplectic structure $\omega_{can} = d\lambda_{can}$.

We want to find the Hamiltonian flow corresponding to the geodesic flow on (N, g). Thus we will derive the Hamiltonian function $H : T^*N \to \mathbb{R}$ corresponding to the above Lagrangian $L(q, v) = \frac{1}{2}g_q(v, v)$. We apply the

Legendre transformation

$$H(q,p) = \sup_{v \in T_q N} \left(\langle p, v \rangle - L(q,v) \right)$$

to L in order to obtain the corresponding Hamiltonian H. One can easily see that the supremum is attained at $v(p) \in T_q N$ determined by $p = d_v L(q, v(p)) = \iota_{v(p)} g_q$. There exists a unique $v(p) \in T_q N$ satisfying $p = \iota_{v(p)} g_q$ for each $p \in T_q^* N$ by the nondegeneracy of Riemannian metric. Then we have the Hamiltonian function

$$H(q,p) = \langle p, v(p) \rangle - L(q, v(p)) = g_q(v(p), v(p)) - \frac{1}{2} g_q(v(p), v(p)) = \frac{1}{2} g_q^*(p,p)$$

on T^*N where g^* is the metric on T^*N which is dual to g. Intuitively, one can think of a geodesic as a free motion of a particle and hence the Hamiltonian corresponding to the geodesic equation has only the kinetic energy term. We summarize the above discussion in the following Theorem.

Theorem 2.3.1. *Let (N, g) be a Riemannian manifold. Then the Hamiltonian flow of the Hamiltonian*

$$H_g(q,p) = \frac{1}{2} g_q^*(p,p)$$

on the symplectic manifold $(T^*N, d\lambda_{can})$ is a lift of the geodesic flow of (N, g). Namely, the projection $\pi(\phi_H^t((q,p)))$ of a Hamiltonian flow into the base manifold N is the geodesic flow starting at q with tangent vector $v(p) \in T_q N$.

Any Riemannian manifold (N, g) can be interpreted as a Hamiltonian equation on its cotangent bundle T^*N. Let us see the following familiar example.

Example 2.3.2. We consider the 2-sphere (S^2, g_0) with the round metric. Then the Hamiltonian H_S for the geodesic flow on (S^2, g_0) is given by $H_S(q,p) = \frac{1}{2} g_0^*(q)(p,p)$ for $(q,p) \in T^*S^2$. Consider this Hamiltonian in a local coordinate chart. In particular, if we think of the stereographic projection

$$\phi : \mathbb{R}^2 \to S^2 - \{N\}$$

from the north pole $N = (0,0,1)$. The chart map is given by

$$\phi(x_1,x_2) = (\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1}),$$
$$\phi^{-1}(q_1,q_2,q_3) = (\frac{q_1}{1-q_3}, \frac{q_2}{1-q_3}).$$

This induces the canonical local coordinate chart

$$\Phi : T^*\mathbb{R}^2 \to T^*(S^2 - \{N\}) \tag{2.3.1}$$

of the cotangent bundle. The Hamiltonian H on this coordinate chart have the expression

$$\tilde{H}(x,y) := H_S \circ \Phi(x,y) = \frac{1}{8}(|x|^2+1)^2|y|^2$$

in terms of $(x,y) \in T^*\mathbb{R}^2$. This Hamiltonian system defined on $T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ is equivalent to the geodesic problem on $(S^2 - \{N\}, g_0)$.

Of course, not every Hamiltonian system is a geodesic problem. For instance, time-reversibility does not hold for Hamiltonian systems in general. However, Moser in [40] found a beautiful connection between the n-dimensional Kepler problem and the geodesic problem on (S^n, g_{round}). The Kepler problem is important in celestial mechanics as the most fundamental problem. As one knows, the geodesic problem on the standard unit sphere equipped with the round metric is also one of the most fundamental problems in geodesic problems.

We have discussed geodesic with two fixed end points. We can introduce various boundary conditions and these boundary conditions can be translated into Hamiltonian language. An interesting boundary condition is the *periodic boundary condition*. We define the *free loop space* $\Lambda N := C^\infty(S^1, N)$ *of* N and the *energy functional*

$$\mathcal{E} : \Lambda N \to \mathbb{R}, \quad \mathcal{E}(\gamma) = \int_{S^1} \frac{1}{2} g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt$$

on the free loop space ΛN. Then the critical loops are closed geodesics. From

Theorem 2.3.1, closed geodesics will correspond to the periodic orbits of the Hamiltonian flow of the Hamiltonian $H_g(q,p) = \frac{1}{2}g_q^*(p,p)$.

We explain another interesting case which generalize the fixed end points condition. Let Q_0 and Q_1 be submanifolds of N. We define the space

$$\mathcal{P}_{Q_0,Q_1}(N) := \{\gamma \in C^\infty([0,1],N) | \gamma(0) \in Q_0, \gamma(1) \in Q_1\}$$

of paths from Q_0 to Q_1 on N and the energy functional

$$\mathcal{E} : \mathcal{P}_{Q_0,Q_1}(N) \to \mathbb{R}, \quad \mathcal{E}(\gamma) = \int_0^1 \frac{1}{2} g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt.$$

From the variational calculus in [37], we have the *differential*

$$d\mathcal{E}(\gamma)[\xi] = -g_{\gamma(1)}(\dot\gamma(1), \xi(1)) + g_{\gamma(0)}(\dot\gamma(0), \xi(0)) - \int_0^1 g_{\gamma(t)}(\nabla_{\dot\gamma}\dot\gamma(t), \xi(t))dt$$

of $\mathcal{E}$ *for* $\xi \in T_\gamma \mathcal{P}_{Q_0,Q_1}(N)$. Therefore, if γ is a critical path, then we should have $\dot\gamma(i) \perp T_{\gamma(i)}Q_i$ in order to make the boundary terms vanish. This condition gives a Lagrangian boundary condition called *conormal boundary condition.*

Let $\pi : T^*N \to N$ be the cotangent bundle of a manifold N.

Definition 2.3.3. Let Q be a smooth submanifold of N. The subbundle

$$\nu^*Q := \{x \in T^*N | \pi(x) \in Q, x(v) = 0 \text{ for all } v \in T_{\pi(x)}Q\}$$

of $T^*N|_Q$ is called the *conormal bundle of* Q.

Example 2.3.4. If Q is the zero section N, then ν^*Q is the zero section N. If Q is a single point $q \in N$, then $\nu^*Q = T_q^*N$. Note that ν^*Q is Lagrangian submanifold in any case.

In fact, every conormal bundle is Lagrangian submanifold of T^*N. It is easy to see that $\lambda_{can}|_{\nu^*Q} = 0$. Moreover, one can find a proof of the following Proposition in [2].

Proposition 2.3.5. *For an n-dimensional submanifold L of T^*N which is a closed subset of T^*N, $\lambda_{can}|_L = 0$ if and only if $L = \nu^*Q$ for $Q = L \cap N$ where N means the zero section of T^*N.*

We have the following Theorem.

Theorem 2.3.6. *Assume that Q_0, Q_1 are submanifolds of a Riemannian manifold (N, g). Let $\mathcal{E}$ be the energy functional on $\mathcal{P}_{Q_0,Q_1}(N)$. Then there is one-to-one correspondence between the set of critical points $Crit(\mathcal{E})$ and the set of Hamiltonian paths $x : [0,1] \to T^*N$ satisfying*

$$\dot{x}(t) = X_{H_g}(x(t)), \quad x(0) \in \nu^*Q_0, \quad x(1) \in \nu^*Q_1$$

for $H_g(q,p) = \frac{1}{2} g_q^*(p,p)$.

Proof. Since geodesic flows are lifted to Hamiltonian flows of H_g by Theorem 2.3.1, it suffices to show the boundary condition. If $\gamma \in Crit(\mathcal{E})$ is a critical path, then the corresponding Hamiltonian path $x = (q,p)$ is given by

$$q(t) = \gamma(t), \quad p(t) = \iota_{\dot{\gamma}(t)} g_{\gamma(t)}$$

and we have to show $x(0) \in \nu^*Q_0$ and $x(1) \in \nu^*Q_1$. We showed that $\dot{\gamma}(i) \perp T_{\gamma(i)}Q_i$ for $i = 0, 1$. This implies that $p(i) = \iota_{\dot{\gamma}(i)} g_{\gamma(i)}$ vanishes on $T_{\gamma(i)}Q_i$ for each $i = 0, 1$. Since $q(i) = \gamma(i) \in Q_i$, we have

$$x(i) = (q(i), p(i)) \in \nu^*Q_i$$

for $i = 0, 1$. This proves the Theorem 2.3.6. □

In [2], they proved that the correspondence in Theorem 2.3.6 gives an isomorphism between Morse homology of $\mathcal{E}$ and Floer homology of the action functional $\mathcal{A}_H$ which will be defined later. In fact, they constructed isomorphisms under the wider class of Lagrangian function and its Legendre transformation. In this thesis, this will play an important role to define symplectic capacity for fiberwise star-shaped domains in a cotangent bundle.

2.4 Contact geometry and Liouville domain

We begin with the definition of the contact manifold.

Definition 2.4.1. Let Σ be a $2n-1$-dimensional smooth manifold. A 1-form $\lambda \in \Omega^1(\Sigma)$ is called a *contact form* if $\lambda \wedge (d\lambda)^{n-1}$ defines a volume form on Σ. We call the pair (Σ, λ) a *contact manifold*. The hyperplane distribution $\xi = \ker \lambda$ is called the *contact structure*.

Remark 2.4.2. One can define the contact manifold without contact form using the notion of contact structure: maximally non-integrable codimension 1 distribution. In fact, this definition using hyperplane distribution is more general notion. Sometimes we will refer a contact manifold by the pair (Σ, ξ) of a manifold and a contact structure. However, we are mostly interested in the case having a contact form defining ξ, because in this thesis we will study the Reeb dynamics defined below. The study of contact structures itself is nowadays a deep and interesting topic. One can find this, for example, in the book [25].

Let $(\Sigma, \xi = \ker \lambda)$ be a contact manifold. We define the *Reeb vector field* R_λ *associated to the contact form* λ by

$$\lambda(R_\lambda) = 1, \quad \iota_{R_\lambda} d\lambda = 0.$$

If a path $x : I \to \Sigma$ satisfies the differential equation

$$\frac{d}{dt}x(t) = R_\lambda(x(t)),$$

then we call x a *Reeb orbit*. In this thesis, we are interested in the dynamics of Reeb vector fields. First we consider periodic Reeb orbits. Let $x : [0, T] \to \Sigma$ be a T-periodic Reeb orbit, namely $x(0) = x(T)$. We say that x is *nondegenerate* if $d\phi_\lambda^T(x(0))|_\xi$ does not have an eigenvalue 1. Here, ϕ_λ^t is the *Reeb flow*: one parameter group of diffeomorphisms generated by R_λ. We say that (Σ, λ) is *nondegenerate* if every periodic Reeb orbit is nondegenerate. Nondegeneracy is a generic condition for closed contact manifolds. More precisely, we borrow the following result in [47].

Theorem 2.4.3. *Assume that* (Σ, λ) *is a closed contact manifold. Then there is a subset* $\mathcal{G} \subset C^\infty(\Sigma, \mathbb{R}^+)$ *which is a countable intersection of open and dense subsets of* $C^\infty(\Sigma, \mathbb{R}^+)$ *such that* $(\Sigma, f\lambda)$ *is nondegenerate for all* $f \in \mathcal{G}$.

Thanks to Theorem 2.4.3, we can assume a closed contact manifold (Σ, λ) is nondegenerate if we are allowed to have small perturbations. We denote by $\mathcal{P}(\Sigma, \lambda)$ the *set of all periodic Reeb orbits of* (Σ, λ). The subset $Spec(\Sigma, \lambda)$ of $\mathbb{R}$ denotes the *set of all periods of periodic Reeb orbits.* We note that if (Σ, λ) is nondegenerate, then $Spec(\Sigma, \lambda)$ is a discrete subset of $\mathbb{R}$.

As the analogue of Lagrangian submanifold, we can define Legendrian submanifolds in a contact manifold.

Definition 2.4.4. An $(n-1)$-dimensional submanifold L of a contact manifold (Σ, ξ) is called *Legendrian submanifold* if $T_pL \subset \xi_p$ for all $p \in L$.

Second we consider Reeb orbits connecting given Legendrian submanifolds. Let L_0 and L_1 be Legendrian submanifolds of $(\Sigma, \xi = \ker \lambda)$. They are not necessarily different. If a path $x : [0, T] \to \Sigma$ satisfies

$$\frac{d}{dt}x(t) = R_\lambda(x(t)), \quad x(0) \in L_0, \quad x(T) \in L_1,$$

then x is called a *Reeb chord from* L_0 *to* L_1 and T is called the *action(or length) of* x. We say that the Reeb chord $x : [0, T] \to \Sigma$ is *nondegenerate* if it satisfies $d\phi_\lambda^T(T_{x(0)}L_0) \pitchfork T_{x(T)}L_1$ in Σ. Note that tangent spaces T_xL_i of Legendrian submanifolds are Lagrangian subspaces of the contact plane $(\xi_x, d\lambda_x)$ by definition and the linearized map $d\phi_\lambda^t$ of Reeb flow preserves the contact plane and defines symplectic linear map. Thus we have $d\phi_\lambda^T(T_{x(0)}L_0)$ and $T_{x(T)}L_1$ are Lagrangian subspaces of $(\xi_{x(T)}, d\lambda_{x(T)})$. Let $\mathcal{P}(\Sigma, \lambda; L_0, L_1)$ be the *set of all Reeb chords from* L_0 *to* L_1. If every $x \in \mathcal{P}(\Sigma, \lambda; L_0, L_1)$ is nondegenerate, then we call that the pair of Legendrian submanifolds L_0, L_1 is *nondegenerate*. Nondegenerate condition for the pair of Legendrian submanifolds is a generic condition, namely one can achieve this condition by arbitrarily small perturbation of contact form. We denote by $Spec(\Sigma, \lambda; L_0, L_1) \subset \mathbb{R}$ the *set of all lenths of Reeb chords from* L_0 *to* L_1. For a nondegenerate pair of Legendrian submanifolds, we have that $Spec(\Sigma, \lambda; L_0, L_1)$ is a discrete set.

In particular, Σ can arise as a level set of Hamiltonian. In this case, the Reeb dynamics is equivalent to the Hamiltonian dynamics. We shall see the condition for this case and the relation between Reeb vector fields and Hamiltonian vector fields.

Definition 2.4.5. Let (M, ω) be a symplectic manifold. If a vector field Y satisfies $L_Y \omega = \omega$, then Y is called a *Liouville vector field of* (M, ω). Let $\Sigma \subset M$ be a hypersurface: codimension 1 submanifold of M. If there is a Liouville vector field Y defined in a neighborhood of Σ such that Y is transverse to Σ, then Σ is called a *contact type hypersurface*.

Proposition 2.4.6. *Assume that (M, ω) be a $2n$-dimensional sympelctic manifold. If a hypersurface Σ is of contact type with transverse Liouville vector field Y defined in a neighborhood of Σ, then $(\Sigma, \lambda := \iota_Y \omega|_\Sigma)$ is a contact manifold. Furthermore, the Reeb vector field is a section of the canonical line bundle $L_\Sigma \to \Sigma$.*

Proof. Since $L_Y \omega = \omega$, we have $d\iota_Y \omega = \omega$ by Cartan formula and so $d\lambda = \omega|_\Sigma$. Therefore, we have that

$$\lambda \wedge (d\lambda)^{n-1} = \iota_Y \omega \wedge \omega^{n-1}|_\Sigma = \frac{1}{n} \iota_Y \omega^n|_\Sigma.$$

Since Y is transverse to Σ, $\{Y, v_1, \cdots, v_{n-1}\}$ forms a basis for any basis $\{v_1, \cdots, v_{n-1}\}$ of $T\Sigma$. Moreover, we know that ω^n defines a volume form on M. This implies that

$$\frac{1}{n} \iota_Y \omega^n|_\Sigma (v_1, \cdots, v_{n-1}) = \frac{1}{n} \omega^n (Y, v_1, \cdots, v_{n-1}) \neq 0$$

and so $\lambda \wedge (d\lambda)^{n-1}$ defines a volume form on Σ. By the definition of the Reeb orbit R_λ, we have that $\iota_{R_\lambda} d\lambda = \iota_{R_\lambda} \omega|_\Sigma = 0$. This implies that $R_\lambda \in \ker \omega|_\Sigma$. This shows that R_λ defines a section of the canonical line bundle. □

From Proposition 2.4.6 and Lemma 2.1.13, then we obtain the following Corollary.

Corollary 2.4.7. *Let (M, ω) be a symplectic manifold. Assume that a time-independent Hamiltonian $H : M \to \mathbb{R}$ has an energy hypersurface $\Sigma = H^{-1}(c)$ of contact type. Then the Reeb dynamics on $(\Sigma, \iota_Y \omega|_\Sigma)$ is equivalent to the Hamiltonian dynamics on Σ.*

As a special case, Σ could be the boundary of a manifold M and the transverse Liouville vector field could be globally defined. We define this special case.

Definition 2.4.8. A *Liouville domain* is a compact symplectic manifold $(M, \omega = d\lambda)$ with a boundary $\Sigma = \partial M$ having a *Liouville vector field* Y such that Y is pointing outward along Σ and $\iota_Y \omega = \lambda$.

We will see in section 5.1 that star-shaped domains in $\mathbb{R}^{2n}$ and fiberwise star-shaped domains in a cotangent bundle T^*N are Liouville domains. In particular, their symplectic homologies will play an important role in this thesis.

Finally, we consider the real Liouville manifold which will appear in the thesis as important examples for wrapped Floer homology.

Definition 2.4.9. A *real Liouville domain* is a triple $(M, \lambda, \mathcal{R})$ consisting of a Liouville domain (M, λ) and an exact anti-symplectic involution $\mathcal{R}$, that is, a diffeomorphism $\mathcal{R} : M \to M$ satisfying

$$\mathcal{R}^2 = \mathrm{id}, \quad \mathcal{R}^*\lambda = -\lambda.$$

on (M, λ).

Let $(M, \lambda, \mathcal{R})$ be a real Liouville domain. Then its boundary $\Sigma = \partial M$ with the restrictions $\lambda|_\Sigma$ and $\mathcal{R}|_\Sigma$ of λ and $\mathcal{R}$ defines a *real contact manifold.* For notational convenience, we will denote by λ and $\mathcal{R}$, respectively, the restriction of λ and $\mathcal{R}$ to Σ. The Reeb vector field R_λ of λ satisfies the following identity $\mathcal{R}_* R_\lambda = -R_\lambda$ for the push-forward $\mathcal{R}_*$ of $\mathcal{R}$. This identity induces an involution $\mathcal{R}_*$ on the Reeb orbit. If $v \in C^\infty([a, b], \Sigma)$ is a Reeb orbit, then a curve $\mathcal{R}_* v \in C^\infty([a, b], \Sigma)$ defined by

$$\mathcal{R}_* v(t) := \mathcal{R}(v(a + b - t))$$

is a Reeb orbit. We define the symmetric Reeb orbit as the fixed point under this induced involution on the Reeb orbit.

Definition 2.4.10. A T-periodic Reeb orbit $v \in C^\infty([0,T],\Sigma)$ on a real contact manifold $(\Sigma, \lambda, \mathcal{R})$ is called *symmetric* if $v = \mathcal{R}_* v$.

Let $(M, \lambda, \mathcal{R})$ be a real Liouville domain. Assume that the fixed point set Fix$(\mathcal{R})$ of the involution $\mathcal{R}$ is nonempty. Then we have that $\mathcal{L}$ =Fix$(\mathcal{R})$ is a Lagrangian in M. Suppose that $\partial\mathcal{L} = \mathcal{L} \cap \partial M$ defines a Legendrian submanifold of $\Sigma = \partial M$. Let $v : [0,T] \to \Sigma$ be a Reeb chord from $\partial\mathcal{L}$ to itself. Then its involution $\mathcal{R}_* v : [0,T] \to \Sigma$ becomes a Reeb chord. Since the following

$$v(0) = \mathcal{R}_* v(T), \quad v(T) = \mathcal{R}_* v(0)$$

and

$$\frac{d}{dt}v(0) = \frac{d}{dt}(\mathcal{R}_* v)(T), \quad \frac{d}{dt}v(T) = \frac{d}{dt}(\mathcal{R}_* v)(0)$$

hold, the concatenation $v \# \mathcal{R}_* v$ is a well-defined $2T$-periodic Reeb orbit. Conversely, if we have a symmetric T-periodic orbit $\gamma : [0,T] \to M$, then the points $\gamma(0), \gamma(\frac{T}{2})$ lie on $\partial\mathcal{L}$ and so the restriction of γ to $[0, \frac{T}{2}]$ becomes a Reeb chord from $\partial\mathcal{L}$ to $\partial\mathcal{L}$. We have proven the following Theorem.

Theorem 2.4.11. *Let $(\Sigma, \lambda, \mathcal{R})$ be a real contact manifold. Assume that $L = Fix(\mathcal{R})$ is a Legendrian submanifold of Σ. Then there is a one-to-one correspondence between the set of all Reeb chords from L to itself and the set of all symmetric periodic orbits.*

We will see examples of real Liouville domains and their wrapped Floer homologies in section 5.2.

Chapter 3

The restricted three body problem and its limit problems

Since Isaac Newton discovered the law of universal gravitation, the three body problem has been a naturally arising but extremely hard problem. The *three body problem* is the problem determining the motion of three bodies under the gravitational force for given initial positions and velocities of three bodies. This problem is one of the hardest problems in celestial mechanics. For instance, the dimension of the phase space of the three body problem is 18. By the difficulty of the three body problem, one usually gives restrictions to this problem. One problem with reasonable and practical restrictions is the *(planar circular) restricted three body problem.* The planar circular restricted three body problem is a simplified version of the three body problem by considering the motion of only one particle which is regarded massless particle. Additionally we assume that three particles have planar motion and two primaries take Keplerian circular motions in the planar circular restricted three body problem. Throughout this thesis, the restricted three body problem always means the planar circular restricted three body problem. In section 3.1, we will describe the restricted three body problem and will derive the time-independent Hamiltonian. We will derive diametrical limit problems of the restricted three body problem, in section 3.2 and 3.3, as our main ingredients of this thesis. Moreover, we will discuss their properties in section 3.2

and 3.3.

3.1 The restricted three body problem

We consider two massive particles on the plane called *primaries* P_1 *and* P_2. We denote the masses M_1, M_2 of two primaries P_1, P_2. We define the *mass ratio* $\mu = \frac{M_2}{M_1+M_2}$ and assume that two primaries are under the *Keplerian circular motion*

$$P_1(t) = (-\mu \cos t, -\mu \sin t), \quad P_2(t) = ((1-\mu)\cos t, (1-\mu)\sin t)$$

with their center of masses at the origin. We are interested in the motion of a massless particle $S(t) \in \mathbb{R}^2 - \{P_1(t), P_2(t)\}$. Example 2.1.5 tells us that the total energy

$$H_I^\mu(t,q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q-P_2(t)|} - \frac{1-\mu}{|q-P_1(t)|}$$

with suitable normalizations of physical constants defines the *time-dependent Hamiltonian of the restricted three body problem.* We write the subscript I to emphasize that this Hamiltonian is defined in the inertial system. Note that H_I^μ is a time-dependent Hamiltonian and so the value of this Hamiltonian is not conserved under time evolution. In this sense it is useful to get a time-independent Hamiltonian. We equip the rotating system and this rotating system will fix the positions of two primaries. We define two points

$$A_1 := (-\mu, 0), \quad A_2 := (1-\mu, 0)$$

on the plane. Then we have the motion of two primaries

$$P_1(t) = R^t A_1, \quad P_2(t) = R^t A_2$$

in terms of A_1 and A_2 where R^t is the time-dependent linear transformation of the form

$$R^t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

We define $\Psi^t := R^t \oplus R^t$ time-dependent endomorphism on $\mathbb{R}^4$. We showed in Example 2.2.4 that the time-dependent transformation Ψ^t is the Hamiltonian diffeomorphism generated by the Hamiltonian $L := q_1 p_2 - q_2 p_1$ on $(\mathbb{R}^4, dp \wedge dq)$. We prove the following Theorem which provides us the time-independent Hamiltonian of the restricted three body problem.

Theorem 3.1.1. *Let H_R^μ be the Hamiltonian of the restricted three body problem in the rotating system of Ψ^t. Then $H_R^\mu = H_I^\mu \circ \phi_L^t - L$ where $L = q_1 p_2 - q_2 p_1$ and ϕ_L^t are Hamiltonian diffeomorphisms generated by L. In particular the Hamiltonian H_R^μ is time-independent.*

Proof. We omit the superscript μ in the proof. The Hamiltonian flow in the rotating coordinate can be written by $\Psi^{-t} \circ \phi_{H_I}^t$ for all $t \in \mathbb{R}$. We have seen that $\Psi^t = \phi_L^t$ and hence we have

$$\Psi^{-t} \circ \phi_{H_I}^t = \phi_L^{-t} \circ \phi_{H_I}^t = \phi_{-L}^t \circ \phi_{H_I}^t = \phi_{(-L)\#H_I}^t$$

by Lemma 2.1.10. By the definition of '#', we obtain the Hamiltonian of the restricted three body problem

$$H_R(t, q, p) = (-L)\#H_I(t, q, p) = H_I(t, \phi_L^t(q, p)) - L(q, p)$$

in the rotating coordinate system. Note that we use $(\phi_{-L}^t)^{-1} = \phi_L^t$. The last statement follows from the computation

$$\begin{aligned} H_I(t, \phi_L^t(q, p)) &= H_I(t, R^t(q), R^t(p)) \\ &= \frac{1}{2}|R^t(p)|^2 - \frac{\mu}{|R^t(q) - P_2(t)|} - \frac{1-\mu}{|R^t(q) - P_1(t)|} \\ &= \frac{1}{2}|R^t(p)|^2 - \frac{\mu}{|R^t(q) - R^t(A_2)|} - \frac{1-\mu}{|R^t(q) - R^t(A_1)|} \\ &= \frac{1}{2}|p|^2 - \frac{\mu}{|q - A_2|} - \frac{1-\mu}{|q - A_1|} \end{aligned}$$

of $H_I \circ \phi_L^t$ and this is time-independent. $\square$

From Theorem 3.1.1, we have the *time-independent Hamiltonian*

$$H_R^\mu : (\mathbb{R}^2 - \{A_1, A_2\}) \times \mathbb{R}^2 \to \mathbb{R},$$

$$H_R^\mu(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - A_2|} - \frac{1-\mu}{|q - A_1|} - q_1 p_2 + q_2 p_1 \tag{3.1.1}$$

for the restricted three body problem.

As we discussed before, the time-independent Hamiltonian H_R^μ is preserved under the Hamiltonian flow of itself. In fact, the integral $-2H_R^\mu$, called *Jacobi integral*, was discovered first by Jacobi. One can ask if there is another integral for the restricted three body problem and this question was negatively answered by Poincaré. Poincaré proved that there is no real analytic integral which is also analytic in μ except the Jacobi integral. This can be found in his book [44]. This result was extended to the analytic non-integrability of the restricted three body problem for all but finite values of μ by Xia in [53] using the existence of the transversal homoclinic orbits.

Remark 3.1.2. Even though the restricted three body problem has no integral, there is a discrete symmetry of the restricted three body problem.

$$\mathcal{R}_1 : (q_1, q_2, p_1, p_2) \longmapsto (q_1, -q_2, -p_1, p_2)$$

$\mathcal{R}_1$ is an anti-symplectic reflection and Hamiltonian H_R^μ is invariant under $\mathcal{R}_1$, namely $H_R^\mu(\mathcal{R}_1(q,p)) = H_R^\mu(q,p)$ for all $(q,p) \in \mathbb{R}^4$. This anti-symplectic involution will give us the real Liouville domain structure defined in section 2.4 after suitable regularization.

A basic approach for the restricted three body problem is investigating the region of possible positions for each energy level. This is called *Hill's region*. Let us determine Hill's region of the restricted three body problem. We can rewrite the Hamiltonian

$$H_R^\mu(q,p) = \frac{1}{2}((p_1 + q_2)^2 + (p_2 - q_1)^2) - \frac{\mu}{|q - A_2|} - \frac{1-\mu}{|q - A_1|} - \frac{1}{2}|q|^2$$

and we define the so-called *effective potential*

$$U^\mu : \mathbb{R}^2 - \{A_1, A_2\} \to \mathbb{R}, \quad U^\mu(q) := -\frac{\mu}{|q - A_2|} - \frac{1-\mu}{|q - A_1|} - \frac{1}{2}|q|^2,$$

then there is one-to-one correspondence between the critical point of H_R^μ and U^μ because their difference is only degree 2 terms. Precisely, the projection $\pi(q,p) = q$ onto q-coordinate

$$\pi : \mathrm{Crit}(H_R^\mu) \to \mathrm{Crit}(U^\mu)$$

provides the one-to-one correspondence. Thus, it suffices to investigate the critical points of U^μ in order to know the critical points of H_R^μ. The critical points of U^μ is called the *Lagrangian points* and we will show that there are five critical points of U^μ for each $\mu \in (0,1)$ and we denote by $l_1^\mu, l_2^\mu, l_3^\mu, l_4^\mu, l_5^\mu$ this critical points. From the gradient

$$\left(\frac{\mu(q_1 - (1-\mu))}{|q - (1-\mu, 0)|^3} + \frac{(1-\mu)(q_1 + \mu)}{|q + (\mu, 0)|^3} - q_1, \frac{\mu q_2}{|q - (1-\mu,0)|^3} + \frac{(1-\mu) q_2}{|q + (\mu, 0)|^3} - q_2 \right)$$

of U^μ, we have the following cases satisfying $DU^\mu(q_1, q_2) = (0,0)$.

Case 1) $q_2 \neq 0$,

Then we should have $\frac{\mu}{|q-(1-\mu,0)|^3} + \frac{(1-\mu)}{|q+(\mu,0)|^3} - 1 = 0$ and thus we get

$$\begin{aligned} 0 &= \frac{\mu(q_1 - (1-\mu))}{|q - (1-\mu, 0)|^3} + \frac{(1-\mu)(q_1+\mu)}{|q + (\mu,0)|^3} - q_1 \\ &= q_1 \left(\frac{\mu}{|q - (1-\mu,0)|^3} + \frac{(1-\mu)}{|q + (\mu,0)|^3} - 1 \right) \\ &\qquad + \left(\frac{-\mu(1-\mu)}{|q - (1-\mu, 0)|^3} + \frac{\mu(1-\mu)}{|q + (\mu,0)|^3} \right) \\ &= \mu(1-\mu) \left(\frac{1}{|q + (\mu,0)|^3} - \frac{1}{|q - (1-\mu,0)|^3} \right) \end{aligned}$$

and this implies

$$|q + (\mu, 0)| = |q - (1-\mu, 0)|.$$

We plug this into $\frac{\mu}{|q-(1-\mu,0)|^3} + \frac{(1-\mu)}{|q+(\mu,0)|^3} - 1 = 0$, then we obtain $|q + (\mu, 0)| =$

$|q-(1-\mu,0)|=1$. This gives us two critical points, say $l_4^\mu := (\frac{1}{2}-\mu, \frac{\sqrt{3}}{2}), l_5^\mu := (\frac{1}{2}-\mu, \frac{\sqrt{3}}{2})$. One interesting result is that $\{l_4^\mu, A_1, A_2\}$ and $\{l_5^\mu, A_1, A_2\}$ form equilateral triangles, respectively.

Case 2) $q_2 = 0$,

We must solve the following 1-variable equation.

$$\frac{\mu(q_1-(1-\mu))}{|q_1-(1-\mu)|^3} + \frac{(1-\mu)(q_1+\mu)}{|q_1+\mu|^3} - q_1 = 0.$$

We define the function

$$u^\mu : \mathbb{R}\backslash\{-\mu,(1-\mu)\} \to \mathbb{R}, \quad u^\mu(x) := \frac{\mu(x-(1-\mu))}{|x-(1-\mu)|^3} + \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - x.$$

The function u^μ is strictly decreasing on each of intervals $(-\infty,-\mu)$, $(-\mu, 1-\mu)$ and $(1-\mu,+\infty)$. Thus we have three zeros $m_1^\mu, m_2^\mu, m_3^\mu$ such that $-\infty < m_3^\mu < -\mu < m_1^\mu < 1-\mu < m_2^\mu < +\infty$. This gives us three critical points

$$l_1^\mu = (m_1^\mu, 0), \quad l_2^\mu = (m_2^\mu, 0), \quad l_3^\mu = (m_3^\mu, 0)$$

of U^μ.

Therefore, we have all five Lagrangian points l_i^μ, $i = 1, 2, \cdots, 5$. Under the map $(q_1, q_2) \mapsto (q_1, q_2, -q_2, q_1)$, these correspond to the critical points L_i^μ of H_R^μ for $i = 1, 2, \cdots, 5$. We define the critical energy values

$$c_i^\mu := H_R^\mu(L_i^\mu) = U^\mu(l_i^\mu)$$

of the restricted three body problem. We borrow the following statement. One can find the proof with details in [23].

Theorem 3.1.3. *For each $\mu \in (0,1)$, the Hamiltonian H_R^μ of the restricted three body problem has five critical points L_i^μ for $i = 1, 2, \cdots, 5$. The corresponding critical energy levels satisfy*

$$c_1^\mu < c_2^\mu < c_3^\mu < c_4^\mu = c_5^\mu \quad \textit{if} \quad \mu < \frac{1}{2},$$

$$c_1^\mu < c_2^\mu = c_3^\mu < c_4^\mu = c_5^\mu \quad \textit{if} \quad \mu = \frac{1}{2},$$
$$c_1^\mu < c_3^\mu < c_2^\mu < c_4^\mu = c_5^\mu \quad \textit{if} \quad \mu > \frac{1}{2}.$$

Moreover, the Morse indices of the critical points is $ind(L_1^\mu) = ind(L_2^\mu) = ind(L_3^\mu) = 1$ *and* $ind(L_4^\mu) = ind(L_5^\mu) = 2$ *for all* $\mu \in (0,1)$.

3.2 The rotating Kepler problem

If we take the limit $\mu \to 0$ for the Hamiltonian (3.1.1) of the restricted three body problem, then we can obtain the *Hamiltonian*

$$H_{RKP} : (\mathbb{R}^2 - \{(0,0)\}) \times \mathbb{R}^2 \to \mathbb{R},$$
$$H_{RKP}(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} - q_1p_2 + q_2p_1 \tag{3.2.1}$$

of the rotating Kepler problem. By the discussion in Example 2.2.4, $\{H_{KP}, L\} = 0$ and so $(-L)\#H_{KP} = H_{KP} - L$. This implies that the rotating Kepler problem is nothing but the planar Kepler problem in the rotating coordinate system. Moreover, the Hamiltonian $H_{RKP} = H_{KP} - L$ satisfies

$$\{H_{RKP}, L\} = \{H_{KP}, L\} - \{L, L\} = 0.$$

Thus the angular momentum L is an integral of the rotating Kepler problem. Therefore, the rotating Kepler problem is a completely integrable system. Using the argument in Theorem 3.1.1, we obtain

$$\phi^t_{H_{RKP}} = \Psi^{-t} \circ \phi^t_{H_{KP}}.$$

Thus it is sufficient to solve the Kepler problem in order to solve the rotating Kepler problem. We will see in section 4.1 that every orbit of the Kepler problem is periodic for each energy below the critical level. The circular orbits of the Kepler problem always can be lifted to the periodic orbits of the rotating Kepler problem. But the non-circular elliptic periodic orbits become the periodic orbits of the rotating Kepler problem only when some conditions hold. The periodic orbits of the rotating Kepler problem will be discussed in section 7.1.

The rotating Kepler problem has only one critical value and this value corresponds to S^1-family of critical points due to the rotation symmetry. We can rewrite the Hamiltonian

$$H_{RKP}(q,p) = \frac{1}{2}((p_1+q_2)^2+(p_2-q_1))^2+U_{RKP}(q), \quad U_{RKP}(q) = -\frac{1}{|q|}-\frac{1}{2}|q|^2$$

using the *effective potential* U_{RKP}. As in the restricted three body problem case, the critical points of H_{RKP} correspond bijectively to the critical points of U_{RKP} via projection onto q-coordinate. It is easy to see that

$$\mathrm{Crit}(U_{RKP}) = \{|q| = 1\}$$

and we denote by

$$-c_R^0 = -\frac{3}{2} = H_{RKP}(q_1, q_2, -q_2, q_1) \quad \text{for} \quad |q| = 1$$

the *critical value of the rotating Kepler problem.*

3.3 Hill's lunar problem

Hill's lunar problem is another limit problem of the restricted three body problem. Hill's lunar problem was derived by Hill in [27] for studying the motion of the Moon. If we consider the lunar theory in the Sun-Earth-Moon system, then we can regard the Moon as the massless particle of the restricted three body problem. The lunar theory is a limit case of the restricted three body problem in the following sense. First, the Sun is much heavier than the Earth. Second, the distance between the Sun and the Moon is much longer than the distance between the Earth and the Moon. In modern language, Hill's idea is taking a blow up of the coordinates near the Earth to the power $\frac{1}{3}$ of μ when one takes $\mu \to 0$. We will borrow the simple derivation from [36].

This will show the relation between Hill's lunar problem and the restricted three body problem. We apply the coordinate change on the Hamiltonian H_R^μ by applying translation on (q,p)-coordinates in the following way.

$$q_1 \to q_1 + 1 - \mu, \quad q_2 \to q_2, \quad p_1 \to p_1, \quad p_2 \to p_2 + 1 - \mu.$$

The Hamiltonian (3.1.1) of the restricted three body problem becomes

$$H_3^\mu(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q|} - \frac{1-\mu}{|q+(1,0)|} - q_1p_2 + q_2p_1 - (1-\mu)q_1$$

up to constant. By Newton's binomial series $(1+x)^{\frac{-1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$, we get the expansion

$$-\frac{1-\mu}{\sqrt{(q_1+1)^2+q_2^2}} = -(1-\mu)(1 - q_1 + q_1^2 - \frac{1}{2}q_2^2 + \cdots)$$

and we apply this on H_3^μ

$$H_3^\mu(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q|} - q_1p_2 + q_2p_1 - (1-\mu)(q_1^2 - \frac{1}{2}q_2^2 + \cdots).$$

We consider the scaling $q \to \mu^{\frac{1}{3}}q, p \to \mu^{\frac{1}{3}}p$ which is symplectic with conformal coefficient $\mu^{\frac{-2}{3}}$. We multiply this factor

$$\mu^{\frac{-2}{3}}H_3^\mu(\mu^{\frac{1}{3}}q, \mu^{\frac{1}{3}}p) = H_{HLP}(q,p) + O(\mu^{\frac{1}{3}}),$$

then we have the *Hamiltonian*

$$H_{HLP} : (\mathbb{R}^2 - \{(0,0)\}) \times \mathbb{R}^2 \to \mathbb{R},$$

$$H_{HLP}(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} - q_1p_2 + q_2p_1 - q_1^2 + \frac{1}{2}q_2^2 \tag{3.3.1}$$

of Hill's lunar problem by taking $\mu \to 0$.

Hill's lunar problem is non-integrable like the restricted three body problem. The non-integrability of Hill's lunar problem has been determined in many versions. Meletlidou, Ichtiaroglou and Winterberg in [35] proved the analytic non-integrability of Hill's lunar problem. Morales-Ruiz, Simó and Simon gave an algebraic proof of the meromorphic non-integrability in [39]. Recently, Llibre and Roberto [33] discussed the C^1-integrability based on the existence of two periodic orbits. Since Hill's lunar problem is non-integrable, Hill's lunar problem provides a good non-integrable test ground for the restricted three body problem.

Remark 3.3.1. There are discrete symmetries of Hill's lunar problem.

$$\mathcal{R}_1 : (q_1, q_2, p_1, p_2) \longmapsto (q_1, -q_2, -p_1, p_2),$$
$$\mathcal{R}_2 : (q_1, q_2, p_1, p_2) \longmapsto (-q_1, q_2, p_1, -p_2).$$

These are anti-symplectic reflections and Hamiltonian H_{HLP} is invariant under these maps.

We can write the Hamiltonian

$$H_{HLP}(q,p) = \frac{1}{2}((p_1+q_2)^2 + (p_2-q_1)^2) + U_{HLP}(q), \quad U_{HLP}(q) := -\frac{1}{|q|} - \frac{3}{2}q_1^2$$

using the *effective potential* U_{HLP}. Following the argument in section 3.1 and 3.2, the critical points and value

$$\mathrm{Crit}(U_{HLP}) = (\pm 3^{\frac{-1}{3}}, 0), \quad U_{HLP}(\pm 3^{\frac{-1}{3}}, 0) = -\frac{3^{\frac{4}{3}}}{2} =: -c_H^0.$$

of U_{HLP} gives the *critical points*

$$Crit(H_{HLP}) = \{(3^{\frac{-1}{3}}, 0, 0, 3^{\frac{-1}{3}}), (-3^{\frac{-1}{3}}, 0, 0, -3^{\frac{-1}{3}})\} \in \mathbb{R}^2(q) \times \mathbb{R}^2(p)$$

and the critical value $H_{HLP}((\pm 3^{\frac{-1}{3}}, 0, 0, \pm 3^{\frac{-1}{3}})) = -c_H^0$ *of* H_{HLP}.

Chapter 4

Moser regularization

In this chapter, we will see the Moser regularization for the Kepler problem in section 4.1. This will show us the correspondence between the Kepler problem and the geodesic problem on the standard sphere. We will discuss its generalizations: fiberwise convexity in section 4.2 and fiberwise star-shapedness in section 4.3, respectively.

4.1 Kepler problem and Moser regularization

We will discuss only the planar case. One can find the proof for general case in [40]. Recall the Hamiltonian of the planar Kepler problem

$$H_{KP} : T^*(\mathbb{R}^2 - \{(0,0)\}) \to \mathbb{R}, \quad H_{KP}(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

from Example 2.2.4. We define the Hamiltonian

$$K_{KP}^c(q,p) = |q|(H_{KP}(q,p) + c) = \frac{1}{2}(|p|^2 + 2c)|q| - 1$$

in order to remove the singularity. Note that the energy hypersurfaces are same, i.e., $(K_{KP}^c)^{-1}(0) = H_{KP}^{-1}(-c)$, and so their Hamiltonian vector fields $X_{K_{KP}^c}$ and $X_{H_{KP}}$ are parallel on this common energy hypersurfaces $(K_{KP}^c)^{-1}(0)$ by Lemma 2.1.13. We focus on the case of $c = \frac{1}{2}$. Other negative energy levels can be achieved simply by rescaling the variables and we will explain this later. We consider the following symplectic transformation

$$\Psi : (T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2, dy \wedge dx) \to (T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq),$$

$$\Psi(x, y) = (y, -x), \tag{4.1.1}$$

namely $p = -x$, $q = y$. Then we define the Hamiltonian

$$\tilde{K}(x, y) := K_{KP}^{\frac{1}{2}} \circ \Psi = \frac{1}{2}(|x|^2 + 1)|y| - 1$$

by applying the above symplectic transformation. We remark that this symplectic transformation plays the role of changing the position and momentum variables in our case. We recall the Hamiltonian $\tilde{H}(x, y) = \frac{1}{8}(|x|^2 + 1)^2|y|^2$ on $T^*\mathbb{R}^2$ in Example 2.3.2. Then the energy hypersurfaces $\tilde{K}^{-1}(0)$ and $\tilde{H}^{-1}(\frac{1}{2})$ are same and so they have same Hamiltonian flows up to reparametrization. We know these energy hypersurfaces $\tilde{K}^{-1}(0)$ and $\tilde{H}^{-1}(\frac{1}{2})$ come from $H_{KP}^{-1}(-\frac{1}{2})$ and $H_S^{-1}(\frac{1}{2})$, respectively, where $H_S(q, p) = \frac{1}{2}g^*_{round}(p, p)$ for each $(q, p) \in T^*S^2$. Thus we have partially proven the following Theorem.

Theorem 4.1.1 (Moser). *For a negative energy $c < 0$, the energy hypersurface $H_{KP}^{-1}(c)$ can be symplectically embedded into the cotangent bundle T^*S^2 as the hypersurface $\{(q, p) \in T^*(S^2 - \{N\}) | g^*_{round}(p, p) = -2c\}$. Moreover, we can compactify these energy hypersurfaces into $\{(q, p) \in T^*S^2 | g^*_{round}(p, p) = -2c\}$ by adding the collision orbits.*

Let us rewrite what we have done. Using the composition of maps (2.3.1) and (4.1.1)

$$(T^*\mathbb{R}^2, \omega_{std}) \xrightarrow[Symp.]{\Psi} (T^*\mathbb{R}^2, \omega_{std}) \xrightarrow[Stereo.]{\Phi} (T^*S^2, \omega), \tag{4.1.2}$$

we embed $H_{KP}^{-1}(-\frac{1}{2})$ into T^*S^2 symplectically. Then the closure of image $\overline{\Phi \circ \Psi(H_{KP}^{-1}(-\frac{1}{2}))}$ under map (4.1.2) is the unit cotangent bundle $S_1^*S^2$ of (S^2, g_{round}). Moreover, the limit points newly come in the closure correspond to the unit circle fiber at the north pole and physically correspond to the collision orbits. In general, if we choose another energy level, then we have a hypersurface

$$\overline{\Phi \circ \Psi(H_{KP}^{-1}(-c))} = \Sigma_K^c \subset T^*S^2$$

of the cotangent bundle over S^2. This hypersurface can be interpreted as a unit cotangent bundle of S^2 with respect to a Riemannian metric g_c. We have many possibilities for a generalization. For example, one can replace Ψ and Φ by other symplectomorphisms. Consider the symplectic linear map $T_c : T^*\mathbb{R} \to T^*\mathbb{R}$, $T_c(q,p) = (\frac{q}{\sqrt{2c}}, \sqrt{2c}p)$. If we replace Ψ by $\Psi \circ T_c$, then we have that

$$\overline{\Phi \circ \Psi \circ T_c(H_{KP}^{-1}(-c))} = S^*_{\sqrt{2c}}S^2 := \{(q,p) \in T^*S^2 | \sqrt{g_0^*(q)(p,p)} = \sqrt{2c}\}$$

for each $c > 0$. In fact, this completes the proof of Theorem 4.1.1 for the planar Kepler problem.

4.2 Fiberwise convexity

In the last part of section 4.1, we discussed possibilities of generalization of Moser regularization from replacing Ψ by another symplectomorphism. In this section, we will discuss another generalization by considering different metrics on S^2.

Definition 4.2.1. A *Finsler manifold* is a differentiable manifold N equipped with a *Finsler function* F on the tangent bundle TN. Namely, F satisfies the following conditions.

- F is smooth on $TN \backslash N$. Here, N means the zero section.
- $F((q,v)) \geq 0$ for all $(q,v) \in TN$ and $F((q,v)) = 0$ if and only if $v = 0$.
- $F((q, \lambda v)) = \lambda F((q,v))$ for all $\lambda \geq 0$ and $(q,v) \in TN$.
- $F((q, v+w)) \leq F((q,v)) + F((q,w))$ for all $(q,v),(q,w) \in TN$

We call F a *Finsler metric on* N.

Let us define the corresponding geometric object.

Definition 4.2.2. Let N be a differentiable manifold. A hypersurface Σ, codimension 1 submanifold, of the tangent bundle TN is called *fiberwise convex* if $\Sigma \cap T_qN$ bounds a strictly convex bounded domain of T_qN which contains the origin for each $q \in N$.

Remark 4.2.3. There is a one-to-one correspondence

$$\{\text{Finsler metric on } N\} \longleftrightarrow \{\text{Fiberwise convex hypersurface of } TN\},$$
$$F \longmapsto F^{-1}(1).$$

between the set of all Finsler metrics and the set of all fiberwise convex hypersurfaces for any fixed manifold N.

Remark 4.2.4. We can rewrite the above two definitions for the cotangent bundle T^*N by the exactly same way. Moreover, we also have a one-to-one correspondence between the set of dual Finsler metrics on N and the set of fiberwise convex hypersurfaces of T^*N.

We extend the idea of the Moser regularization. Let U be an open subset of $\mathbb{R}^2$. For a given Hamiltonian $H : T^*U \to \mathbb{R}$, we define a subset

$$\Sigma^c := \Phi \circ \Psi(H^{-1}(-c)) \subset T^*S^2$$

of T^*S^2 using the maps (2.3.1) and (4.1.1). If its closure $\overline{\Sigma^c}$ in T^*S^2 is a fiberwise convex hypersurface of T^*S^2, then the Hamiltonian flow on $H^{-1}(-c)$ can be interpreted as a geodesic flow of the corresponding Finsler metric on S^2. In this case, we say that the Hamiltonian system defined by H is *fiberwise convex for energy* $-c$. If we prove fiberwise convexity of a Hamiltonian system, then we can apply the results in the Finsler geometry on S^2. For example, we can prove the existence of two geometrically different periodic orbits by Bangert and Long's result [11] for Finsler 2-spheres. One can also obtain a systole bound using the universal systole bound on S^2 with Finsler metric proven in [9]. Moreover, we can identify the Conley-Zehnder index of a

Hamiltonian periodic orbit with the Morse index of the corresponding closed geodesic. Thus, we can conclude that every periodic orbit, including collision periodic orbits, has nonnegative Conley-Zehnder index.

In [15], Cieliebak, Frauenfelder and van Koert proved fiberwise convexity of the rotating Kepler problem. Fiberwise convexity of Hill's lunar problem was also proven in [32]. One goal in this direction is determining the fiberwise convexity of the restricted three body problem. We still do not know the fiberwise convexity of the restricted three body problem. As our main ingredient, we give the precise statements below.

Theorem 4.2.5 (Fiberwise convexity of the rotating Kepler problem)**.** *The bounded component of the regularized rotating Kepler problem is fiberwise convex for all energy below the critical level.*

Theorem 4.2.6 (Fiberwise convexity of Hill's lunar problem)**.** *The bounded component of the regularized Hill's lunar problem is fiberwise convex for all energy below the critical level.*

Recall the Hamiltonians

$$H_R(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} + p_1q_2 - p_2q_1, \quad H_H(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} + p_1q_2 - p_2q_1 - q_1^2 + \frac{1}{2}q_2^2$$

of the rotating Kepler problem and Hill's lunar problem, respectively. From now on, H_R and H_H will denote the Hamiltonians of the rotating Kepler problem and Hill's lunar problem, respectively. We showed that H_R and H_H have one critical value $-c_R^0 = -\frac{3}{2}$ and $-c_H^0 = -\frac{3^{\frac{4}{3}}}{2}$, respectively. We define the regularized energy hypersurfaces

$$\Sigma_R^c := \overline{(\Phi \circ \Psi(H_R^{-1}(-c)))}^b, \quad \Sigma_H^{c'} := \overline{(\Phi \circ \Psi(H_H^{-1}(-c')))}^b$$

of each problem for some $c > c_R^0$ and $c' > c_H^0$, respectively. The overlines denote the closure in T^*S^2 and superscripts b denote the bounded component in T^*S^2. We will call Σ_R^c(resp. Σ_H^c) the *energy hypersurface of the regularized rotating Kepler problem(resp. Hill's lunar problem) at energy $-c$ for each $c > c_R^0$ (resp. $c > c_H^0$).* The above Theorems mean that Σ_R^c and $\Sigma_H^{c'}$ are fiberwise convex hypersurfaces for each $c > c_R^0$ and $c' > c_H^0$. Therefore, we can regard each of the rotating Kepler problem and Hill's lunar problem as a geodesic problem on S^2 equipped with a Finsler metric. Moreover, fiberwise convexity implies fiberwise star-shapedness and this fact opens the possibility of applying symplectic topology and contact geometry. We will discuss the definition of fiberwise star-shapedness and its result in the next section.

4.3 Fiberwise star-shapedness

We begin with the following definition.

Definition 4.3.1. Let N be a closed smooth manifold. A hypersurface Σ of T^*N is called *fiberwise star-shaped* if $\Sigma \cap T_q^*N$ bounds a star-shaped domain with respect to the origin in T_q^*N for each $q \in N$.

It is obvious from the definition that fiberwise star-shapedness is a weaker property than fiberwise convexity. Fiberwise star-shapedness can be characterized by the property having contact structure.

Proposition 4.3.2. *Suppose that N is a closed smooth n-manifold. A hypersurface Σ of T^*N is a contact manifold with the restriction of the canonical 1-form of T^*N to Σ if and only if Σ is fiberwise star-shaped. Moreover, if Σ is a fiberwise star-shaped hypersurface, then the bounded region enclosed by Σ defines a Liouville domain with the canonical 1-form and the Liouville vector field of T^*N.*

Proof. Assume that Σ is a hypersurface in T^*N. Let λ be the canonical 1-form of T^*N and $\omega = d\lambda$ be the canonical symplectic form of T^*N. Then we have that

$$\lambda|_\Sigma \wedge (d\lambda|_\Sigma)^{n-1} = (\lambda \wedge d\lambda^{n-1})|_\Sigma, \quad \lambda = \iota_Y \omega$$

where Y is the Liouville vector field. Note that $Y = p\frac{\partial}{\partial p}$ in any canonical coordinates system using the equation $\iota_Y dp \wedge dq = pdq$. Thus, by Definition 2.4.1, $\lambda|_\Sigma$ defines a contact form if and only if the $(2n-1)$-form

$$\lambda|_\Sigma \wedge (d\lambda|_\Sigma)^{n-1} = (\iota_Y \omega) \wedge \omega^{n-1}|_\Sigma = \frac{1}{n}\iota_Y(\omega^n)|_\Sigma$$

defines a volume form on Σ if and only if Y is transverse to Σ. Since Y is an outward radial vector on each cotangent space T_q^*N. The $(2n-1)$-form $\lambda|_\Sigma \wedge (d\lambda|_\Sigma)^{n-1}$ defines a volume form on Σ if and only if Σ is a fiberwise star-shaped hypersurface of T^*N. The last statement is clear from Definition 2.4.8. This proves Proposition 4.3.2. □

Following the fiberwise convex case, we define the fiberwise star-shapedness of Hamiltonians. For a given Hamiltonian $H : T^*U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^2$, we embed the energy hypersurface

$$\Sigma^c := \Phi \circ \Psi(H^{-1}(-c))$$

of H at energy $-c$ into the cotangent bundle over S^2 using the maps (2.3.1) and (4.1.1). If its closure $\overline{\Sigma^c}$ in T^*S^2 is a fiberwise star-shaped hypersurface of T^*S^2, then we say that the Hamiltonian H is *fiberwise star-shaped for energy* $-c$. Suppose Σ is a fiberwise star-shaped hypersurface in T^*S^2. Then $(\Sigma, \lambda := \lambda_{can}|_\Sigma)$ is a contact manifold by Proposition 4.3.2. Moreover, by Corollary 2.4.7, the Reeb vector field R_λ on Σ is a section of the canonical line bundle $L_\Sigma \to \Sigma$ and so the Reeb flows are equivalent with the Hamiltonian flows generated by H. As a result, if a Hamiltonian is fiberwise star-shaped at energy $-c$, then the Hamiltonian flow on $H^{-1}(-c)$ can be interpreted as the Reeb flow of corresponding fiberwise star-shaped hypersurface in T^*S^2. First, a fiberwise star-shaped hypersurface is diffeomorphic to $\mathbb{R}P^3$. By Eliashberg' work in [20], there is a unique tight contact structure up to isotopy on $\mathbb{R}P^3$. From the criterion due to Eliashberg and Gromov in [19] and [26], any symplectically fillable contact 3-manifold is tight. Because regularized energy hypersurfaces Σ_R^c and $\Sigma_H^{c'}$ are symplectically fillable and diffeomorphic to $\mathbb{R}P^3$, we have the following Corollary.

Corollary 4.3.3. *The bounded component of the regularized rotating Kepler problem and the regularized Hill's lunar problem has a contact structure for the energy level below each of critical values. Moreover, these contact structures are the unique tight contact structure on $\mathbb{R}P^3$ up to contact isotopy.*

Moreover, for a fiberwise star-shaped hypersurface Σ, the inside M of Σ

in T^*S^2 defines a Liouville domain. We denote by M_R^c and $M_H^{c'}$ the *Liouville domains defined by the regularized energy hypersurfaces of the rotating Kepler problem at energy $-c$ and Hill's lunar problem at energy $-c'$, respectively.* Because the tools in this thesis can be applied to any Liouville domain defined by a fiberwise star-shaped hypersurface in a cotangent bundle. Although the fiberwise convexity of the restricted three body problem is still unknown, it is worthwhile to refer the fiberwise star-shapedness of the restricted three body problem proven in [8]. Furthermore, they proved existence of contact structure slightly above the first Lagrangian value. We recall the result.

Theorem 4.3.4 (Albers-Frauenfelder-van Koert-Paternain)**.** *For a energy c below the first critical value, two bounded components $\Sigma_E^{\mu,c}$ and $\Sigma_M^{\mu,c}$ near the earth and the moon, respectively, of the regularized restricted three body problem in T^*S^2 admit compatible contact forms, respectively. Moreover, there exists $\epsilon > 0$ such that for $-c \in (H_R^\mu(L_1^\mu), H_R^\mu(L_1^\mu)+\epsilon)$ the bounded component $\Sigma_{E,M}^c$ admits a compatible contact form λ.*

Theorem 4.3.4 opened the possibility of using contact topology for the restricted three body problem. We can see that the tight contact structure on $\mathbb{R}P^3 \# \mathbb{R}P^3$ is unique up to isotopy, using the unique decomposition theorem for tight contact structure in [18]. Adding this result, we obtain the following Corollary.

Corollary 4.3.5 (Albers-Frauenfelder-van Koert-Paternain)**.** *For an energy $-c < H_R^\mu(L_1^\mu)$, the contact structures of $(\Sigma_E^c, \ker\lambda_{can})$ and $(\Sigma_M^c, \ker\lambda_{can})$ are the tight contact structure on $\mathbb{R}P^3$ up to contact isotopy. Moreover, for $-c \in (H_R^\mu(L_1^\mu), H_R^\mu(L_1^\mu)+\epsilon)$, the contact structure of $(\Sigma_{E,M}^c, \ker\lambda)$ is the tight contact structure on $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

A recent result of Cristofaro-Gardiner and Hutchings in [17] tells us that every contact 3-manifold has at least two closed Reeb orbits. This general result on contact manifolds tells us again the existence of at least two Hamiltonian periodic orbits for the restricted three body problem and Hill's lunar problem below the first critical energy levels. In particular, this argument valid for the restricted three body problem slightly above the first critical energy level as well.

Chapter 5

Floer homology

In this chapter, we discuss two types of Floer homology. Because our main ingredient of the thesis is Liouville domains defined by fiberwise star-shaped hypersurfaces in a cotangent space T^*N, we need a Floer homology theory for a symplectic manifold with boundary. Fortunately, Floer homology theory for a symplectic manifold with boundary was developed under some assumptions on the boundary. We will consider a simple case: Liouville domains. First, we define the symplectic homology of a Liouville domain in section 5.1. Second, we define the wrapped Floer homology of a Liouville domain with a pair of admissible Lagrangians in section 5.2 under strong assumptions.

5.1 Symplectic homology of Liouville domain

In this thesis, we will use the symplectic homology of cotangent bundles. However, we can define more generally the symplectic homology of Liouville domains without any difference in difficulty. Thus we will define the symplectic homology for Liouville domains. More generally, one can define the symplectic homology for a symplectic manifold (M, ω) with contact type boundary, see Definition 2.4.5, under the following assumptions.

- $(\mathbf{\Omega})$: $[\omega]$ vanishes on $\pi_2(M)$.
- $(\mathbf{C})$: The first Chern class $c_1(M)$ vanishes on $\pi_2(M)$.

One can see this general construction under these assumptions in [13] and [51]. In our case, Liouville domains, assumption ($\mathbf{\Omega}$) always hold by exactness of symplectic form. *Throughout this paper, we will assume that our Liouville domain* $(M, \omega = d\lambda)$ *satisfies assumption* ($\mathbf{C}$). This is necessary to define a integer-valued Conley-Zehnder index. Let us give examples which satisfy ($\mathbf{C}$). In particular, we have to keep in mind Example 5.1.2 throughout the thesis.

Example 5.1.1 (Star-shaped domains in $\mathbb{R}^{2n}$)**.** If we take the unit ball $B_1^{2n}(0) = \{x \in \mathbb{R}^{2n} || x|^2 \leq 1\}$ in $\mathbb{R}^{2n}$ with the symplectic form

$$\omega_0 = d\lambda_0 \quad \text{where} \quad \lambda_0 = \frac{1}{2}(pdq - qdp)$$

is the canonical Liouville form, then a vector field

$$Y_0(q,p) = \frac{1}{2}q\frac{\partial}{\partial q} + \frac{1}{2}p\frac{\partial}{\partial p}$$

is the Liouville vector field. The vector field Y_0 is radial and so it is transverse to the unit sphere $\partial B_1^{2n}(0) = S^{2n-1}$ with outward direction. Thus $(B_1^{2n}(0), \omega_0 = d\lambda_0)$ is a Liouville domain. More generally, if we take a domain $D \in \mathbb{R}^{2n}$ whose boundary $S = \partial D$ is transversal to Y_0, then $(D, \omega_0 = d\lambda_0)$ is a Liouville domain. The condition to have Y_0-transversal boundary is the star-shapedness of D with respect to the origin. Clearly, this example satisfies the condition ($\mathbf{C}$).

Example 5.1.2 (Fiberwise star-shaped domains in T^*N)**.** Let (N, g) be an orientable Riemannian manifold. If we take the unit disk cotangent bundle $D_g^*N = \{x \in T^*N | g^*(x,x) \leq 1\}$ in T^*N with the canonical symplectic form $\omega_{can} = d\lambda_{can}$ where λ_{can} is the canonical 1-form defined in Definition 2.1.6. We have seen that $\lambda_{can} = pdq, \quad \omega_{can} = dp \wedge dq$ in any canonical local coordinate system of T^*N. Thus a vector field $Y_{can} = p\frac{\partial}{\partial p}$ is the Liouville vector field. We proved in Proposition 4.3.2 that any fiberwise star-shaped hypersurface defines a Liouville domain without the notion of metric. By the orientability of N, the condition ($\mathbf{C}$) is satisfied.

Let $(M, \omega = d\lambda)$ be a Liouville domain with Liouville vector field Y. We define the *completion of M* by attaching the *symplectization cylinder* $[1, \infty) \times \partial M$ along ∂M identified with $\{1\} \times \partial M$. Namely, the completion $(\hat{M}, \hat{\omega})$ is an open symplectic manifold

$$\hat{M} = M \cup_{\{1\}\times\partial M} [1, \infty) \times \partial M,$$
$$\hat{\omega} = \begin{cases} \omega & \text{on } M \\ d(r\lambda) & \text{on } [1,\infty)\times\partial M \end{cases}, \quad \hat{\lambda} = \begin{cases} \lambda & \text{on } M \\ r\lambda & \text{on } [1,\infty)\times\partial M \end{cases}$$

where r is the coordinate for the first component $[1, \infty)$ of symplectization cylinder.

Symplectic homology is obtained by taking a limit on a carefully chosen family of Floer homology on $\hat{M}$. First, we will define the Floer homology for 1-periodic Hamiltonians and later we will specify the type of Hamiltonians that we use for the symplectic homology. *Throughout this paper, we will use $\mathbb{Z}_2$-coefficient to avoid the orientation argument.* However, our discussion in this section is still valid in general for $\mathbb{Z}$-coefficient by considering the coherent orientation discussed in [21] and [13].

Let $(\hat{M}, \hat{\omega} = d\hat{\lambda})$ be the completion of a Liouville domain (M, ω). We choose a 1-periodic Hamiltonian $H : S^1 \times \hat{M} \to \mathbb{R}$, $S^1 = \mathbb{R}/\mathbb{Z}$. We define $H_t(x) = H(t, x)$ for notational convenience. We define the *action functional*

$$\mathcal{A}_H(x) = \int_{S^1} x^*\hat{\lambda} - \int_0^1 H(t, x(t))dt \tag{5.1.1}$$

associated to H on the free loop space $\Lambda\hat{M} := C^\infty(S^1, \hat{M})$ of $\hat{M}$. Roughly speaking, we want to formulate Morse homology on $\Lambda\hat{M}$ using the action functional $\mathcal{A}_H$ as a Morse function. Thus, first, we want to know if $\mathcal{A}_H$ is nondegenerate at every critical point. The corresponding concept is the nondegeneracy of Hamiltonians and it is a generic condition as Morse theory.

We observe the critical point of $\mathcal{A}_H$. We compute the differential of $\mathcal{A}_H$

$$\begin{aligned} d\mathcal{A}_H(x)(\hat{v}) &= \int_0^1 -\hat{\omega}(\dot{x}(t),\hat{v}(t)) - dH_t(\hat{v}(t))dt \\ &= \int_0^1 -\hat{\omega}(\dot{x}(t),\hat{v}(t)) + \hat{\omega}(X_H^t(x(t)),\hat{v}(t))dt \\ &= \int_0^1 \hat{\omega}(\hat{v}(t),\dot{x}(t) - X_H^t(x(t)))dt \end{aligned}$$

for a tangent vector $\hat{v} \in T_x\Lambda\hat{M}$ at $x \in \Lambda\hat{M}$ where $\dot{} = \frac{d}{dt}$. Here, we interpret the tangent vector $\hat{v} \in T_x\Lambda\hat{M}$ as a section of pull-back bundle $x^*T\hat{M}$, namely $\hat{v}(t) \in T_{x(t)}\hat{M}$. From this computation, we know that the loop x is a critical point of $\mathcal{A}_H$ if and only if it satisfies the equation of Hamiltonian system

$$\dot{x}(t) - X_H^t(x(t))) = 0 \quad \text{for all} \quad t \in S^1$$

that is, x is a 1-periodic orbit of X_H^t. We have that

$$Crit(\mathcal{A}_H) = \{x \in \Lambda\hat{M} | \dot{x}(t) = X_H^t(x(t))\}$$

and we denote this set by $\mathcal{P}_H$.

Definition 5.1.3. A 1-periodic orbit $x \in \mathcal{P}_H$ is called *nondegenerate* if the linearized Hamiltonian flow of time 1 map

$$d\phi_H^1(x(0)) : T_{x(0)}\hat{M} \to T_{x(0)}\hat{M}$$

at $x(0)$ has no eigenvalue 1, equivalently, $\det(id - d\phi_H^1(x(0))) \neq 0$. A Hamiltonian $H \in C^\infty(S^1 \times \hat{M})$ is called *nondegenerate* if every $x \in \mathcal{P}_H$ is nondegenerate.

Nondegeneracy of Hamiltonian function is a generic condition and we will assume our Hamiltonian H is nondegenerate. If a Hamiltonian H is nondegenerate, then we have well-defined Conley-Zehnder indices for all $x \in \mathcal{P}_H$. The Conley-Zehnder index for paths of symplectic matrices can be found in Appendix A.

We state the definition of the Conley-Zehnder index $\mu_{CZ}(x)$ of a 1-periodic orbit $x \in \mathcal{P}_H$. If x is contractible, then we take a filling disk $\bar{x} : D \to \hat{M}$ and we take a *symplectic trivialization*

$$\bar{\Gamma} : D \times \mathbb{R}^{2n} \to \bar{x}^* T\hat{M}$$

for a symplectic vector bundle $\bar{x}^* T\hat{M} \to D$. This trivialization induces a symplectic trivialization

$$\Gamma : S^1 \times \mathbb{R}^{2n} \to x^* T\hat{M}$$

of the subbundle $x^* T\hat{M} \to S^1$ by restriction. We obtain a path of symplectic matrices

$$\Phi_x^{\Gamma}(t) = \Gamma(t)^{-1} d\phi_H^t(x(0))\Gamma(0) \in Sp(2n), \quad t \in [0,1]$$

from the linearized Hamiltonian flow $d\phi_H^t$. Nondegeneracy of H implies $\Phi_x^{\Gamma} \in SP(2n) := \{\Phi : [0,1] \to Sp(2n) | \Phi(0) = id, \quad \det(\Phi(1) - id) \neq 0\}$. We define the *Conley-Zehnder index of x with respect to $\bar{x}, \bar{\Gamma}$* by

$$\mu_{CZ}(x; \bar{x}, \bar{\Gamma}) := \mu_{CZ}(\Phi_x^{\Gamma}).$$

By the condition (**C**), Conley-Zehnder index of x is independent of the choices of $\bar{x}$ and $\bar{\Gamma}$ and so we will denote simply by $\mu_{CZ}(x) := \mu_{CZ}(x; \bar{x}, \bar{\Gamma})$. For the Conley-Zehnder index of a noncontractible 1-periodic orbit, we choose a representative x_c and a trivialization $\Gamma_c : x_c^* T\hat{M} \to S^1 \times \mathbb{R}^{2n}$ for each $0 \neq c \in H_1(\hat{M}; \mathbb{Z})$. For a given $x \in \mathcal{P}_H$ of $[x] = c$, we extend the trivialization Γ_c along the 2-cycle connecting x_c and x. This induces an trivialization $\Gamma : x^* T\hat{M} \to S^1 \times \mathbb{R}^{2n}$. Then we can define the Conley-Zehnder index of x as before.

One important ingredient of Morse homology is a Riemannian metric. In order to define the metric on $\mathcal{L}\hat{M}$, we introduce the definition of ω-compatible almost complex structure on M.

Definition 5.1.4. Let (M, ω) be a symplectic manifold(possibly with boundary). We call a section $J \in \Gamma(End(TM))$ an *almost complex structure on M* if $J(x)^2 = -id|_{T_xM}$ for all $x \in M$. An almost complex structure J on the symplectic manifold (M, ω) is called *ω-compatible*, if $\omega(\cdot, J\cdot)$ defines a Riemannian metric on M. We denote by $\mathcal{J}_\omega(M)$ the *space of all ω-compatible almost structure.*

We need the following particular type of $\hat{\omega}$-compatible almost complex structure on $\hat{M}$ in order to define symplectic homology.

Definition 5.1.5. Let $(\hat{M}, \hat{\omega} = d\hat{\lambda})$ be a completion of a Liouville domain $(M, \omega = d\lambda)$. An $\hat{\omega}$-compatible almost complex structure J is called *SFT-like* if it satisfies the following conditions

- J preserves the contact hyperplane $\xi = \ker \lambda|_{T\partial M}$ on $(\partial M, \lambda)$.
- $JY = R$ and $JR = -Y$ on ∂M where Y is the Liouville vector field and R is the Reeb vector field.
- J is invariant under the flow of the Liouville vector field Y in the cylindrical end $[1, \infty) \times \partial M$.

Denote by $\mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M})$ the *set of all SFT-like $\hat{\omega}$-compatible almost complex structure on $\hat{M}$.*

The space $\mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M})$ is nonempty and contractible. One can see the proof of this fact, for example, in [34]. We choose an SFT-like $\hat{\omega}$-compatible structure $J \in \mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M})$. From the definition, one can define the metric from J. We denote this metric by $\langle v_1, v_2 \rangle_J := \hat{\omega}(v_1, Jv_2)$ for $v_1, v_2 \in T_p\hat{M}$. Consider a S^1-family of SFT-like almost complex structures $J := \{J_t\}_{t \in S^1}$. This induces a metic on $\Lambda\hat{M}$ by L^2-metic. Let $x \in \Lambda\hat{M}$ be a loop in $\hat{M}$. One can regard a vector of the tangent space $T_x\Lambda\hat{M}$ as a vector field along x, that is, we identify $\hat{v} \in T_x\Lambda\hat{M}$ with a section $\hat{v} \in \Gamma(x^*T\hat{M})$ of pullback bundle. With this identification, we define a metric on $\Lambda\hat{M}$ as follows. Given $\hat{v}_1, \hat{v}_2 \in T_x\Lambda\hat{M} = \Gamma(x^*T\hat{M})$, we define the *$L^2$-metric*

$$\langle\langle \hat{v}_1, \hat{v}_2 \rangle\rangle_J := \int_0^1 \langle \hat{v}_1(x(t)), \hat{v}_2(x(t)) \rangle_{J_t} dt = \int_0^1 \omega_{x(t)}(\hat{v}_1(x(t)), J_t\hat{v}_2(x(t)))dt$$

on $\Lambda\hat{M}$ *associated to* J. We can deduce the gradient flow line equation for $\mathcal{A}_H$ using above computations. Since we have

$$\begin{aligned} d\mathcal{A}_H(x)(\hat{v}) &= \int_0^1 \hat{\omega}(\hat{v}(x(t)), \dot{x}(t) - X_H^t(x(t)))dt \\ &= \langle\langle \hat{v}, -J(\dot{x} - X_H^t)\rangle\rangle_J \end{aligned}$$

for any $\hat{v} \in T_x\Lambda\hat{M}$ and $x \in \Lambda\hat{M}$, we have the *gradient vector*

$$\nabla\mathcal{A}_H(x)(x(t)) = -J_t(x(t))(\dot{x}(t) - X_H^t(x(t)))$$

of $\mathcal{A}_H$ *at* $x \in \Lambda\hat{M}$. This induces the *gradient flow line*

$$u : \mathbb{R} \to \Lambda\hat{M}, \quad \frac{du}{ds} = \nabla\mathcal{A}_H(u(s))$$

of $\mathcal{A}_H$ *on* $\Lambda\hat{M}$. This is an ODE on an infinite dimensional space. Using the identification $C^\infty(\mathbb{R}, \Lambda\hat{M}) = C^\infty(\mathbb{R} \times S^1, \hat{M})$, we can rewrite this ODE to a PDE on $\hat{M}$. Namely, the gradient flow line $u : \mathbb{R} \times S^1 \to \hat{M}$ satisfies the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial s}(s,t) = \nabla\mathcal{A}_H(u(s,t)) \iff \partial_s u + J_t(u)(\partial_t u - X_H^t(u)) = 0 \qquad (5.1.2)$$

called *Floer equation on the infinite cylinder*. As in the Morse homology, we will define the boundary map by counting the gradient flow line. Given $x^\pm \in \mathcal{P}_H$, we denote by $\widehat{\mathcal{M}}(x^-, x^+)$ the *moduli space of gradient flow lines from* x^- *to* x^+, that is,

$$\widehat{\mathcal{M}}(x^-, x^+) = \{u : \mathbb{R} \times S^1 \to \hat{M} | (5.1.2), \lim_{s \to \pm\infty} u(s,t) = x^\pm\}.$$

There is $\mathbb{R}$-action on $\mathbb{R} \times S^1$. We can obtain the unparametrized moduli space by taking quotient by this $\mathbb{R}$-action on $\widehat{\mathcal{M}}(x^-, x^+)$. This quotient is called the *moduli space of Floer trajectories from* x^- *to* x^+ and is denoted by

$$\mathcal{M}(x^-, x^+) := \widehat{\mathcal{M}}(x^-, x^+)/\mathbb{R}.$$

Assume now that all elements in $\mathcal{P}_H$ and the gradient trajectories between them are contained in the compact subset of $\hat{M}$. This will be achieved by taking H with suitable assumptions and we will introduce these assumptions later. For a generic $J \in S^1 \times \mathcal{J}^{SFT}_{\hat{\omega}}(\hat{M})$, the moduli space $\mathcal{M}(x^-, x^+)$ is a smooth manifold of dimension $\mu_{CZ}(x^+) - \mu_{CZ}(x^-) - 1$ for each $x^-, x^+ \in \mathcal{P}_H$. We define the *Floer chain group for H*

$$CF_k^{<a}(H) := \mathbb{Z}_2 \langle x \in \mathcal{P}_H | \mu_{CZ}(x) = k, \mathcal{A}_H(x) < a \rangle$$

as the $\mathbb{Z}_2$-module generated by the 1-periodic orbits of index k for $k \in \mathbb{Z}$ and of action less than a for $a \in \mathbb{R} \cup \{\pm\infty\}$. We abbreviate $CF_k^{<+\infty}(H) = CF_k(H)$. We define the *filtered chain complex*

$$CF_k^{[a,b)}(H) := CF_k^{<b}(H)/CF_k^{<a}(H)$$

for $a < b \in \mathbb{R} \cup \{\pm\infty\}$ and the *boundary map*

$$\partial^{[a,b)}_{(H,J)} : CF_k^{[a,b)}(H) \to CF_{k-1}^{[a,b)}(H), \quad \partial^{[a,b)}_{(H,J)}(x) := \sum_{\substack{y \in \mathcal{P}_H, \\ \mu_{CZ}(y) = k-1, \\ a \leq \mathcal{A}_H(y) < b}} \#_{\mathbb{Z}_2} \mathcal{M}(y, x) y$$

on it. If we have compactness for the moduli spaces, then $\partial^{[a,b)}_{(H,J)}$ is well defined and indeed a boundary map, that is, it satisfies $\partial^{[a,b)}_{(H,J)} \circ \partial^{[a,b)}_{(H,J)} = 0$. Under the compactness assumption, we can define the *filtered Floer homology groups*

$$FH_*^{[a,b)}(H, J) = \ker \partial^{[a,b)}_{(H,J)} / im \partial^{[a,b)}_{(H,J)}$$

for $a < b \in \mathbb{R} \cup \{\pm\infty\}$. From a short exact sequence of chain complexes

$$0 \to CF_*^{[a,b)}(H) \to CF_*^{[a,c)}(H) \to CF_*^{[b,c)}(H) \to 0,$$

we have a long exact sequence of the filtered Floer homology groups

$$\cdots \to FH_*^{[a,b)}(H, J) \to FH_*^{[a,c)}(H, J) \to FH_*^{[b,c)}(H, J) \to FH_{*-1}^{[a,b)}(H, J) \to \cdots \tag{5.1.3}$$

A standard argument in Floer homology theory says that $FH_*^{[a,b)}(H,J)$ is independent of the choice of J. Thus, we denote $FH_*^{[a,b)}(H,J)$ by $FH_*^{[a,b)}(H)$. However, $FH_*^{[a,b)}(H)$ depends on the choice of the Hamiltonian. Moreover, $FH_*^{[a,b)}(H)$ cannot be defined for an arbitrary Hamiltonian due to compactness. We specify the Hamiltonians which guarantee compactness.

Definition 5.1.6. A smooth Hamiltonian $H : S^1 \times \hat{M} \to \mathbb{R}$ is called *admissible* if it satisfies the following conditions

- H is nondegenerate.
- $H|_{S^1 \times M} \leq 0$
- $\lim_{r\to\infty} H(\cdot, r, x) = ar + b$ on symplectic cylinder $(r, x) \in [1, +\infty) \times \partial M$ for some $a, b \in \mathbb{R}$ such that $0 < a \notin Spec(\partial M, \lambda)$.

We denote by $Ad(M)$ the *set of all admissible Hamiltonian on* $\hat{M}$.

For an admissible Hamiltonian $H \in Ad(M)$, there is a S^1-family of SFT-like $\hat{\omega}$-compatible almost complex structure J such that the moduli space $\mathcal{M}(x^-, x^+; H, J)$ is a smooth manifold for each $x^-, x^+ \in \mathcal{P}_H$. Moreover, in fact, the set of all such S^1-family of SFT-like $\hat{\omega}$-compatible almost complex structure forms a Baire set in $C^\infty(S^1, \mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M}))$. We call such pair $(H, J) \in Ad(M) \times C^\infty(S^1, \mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M}))$ an *admissible pair*. We denote by $\mathcal{N}_{reg}(M)$ the *set of all admissible pairs*. For an admissible pair $(H, J) \in \mathcal{N}_{reg}(M)$, we can define the filtered Floer homology $FH_*^{[a,b)}(H)$ for any $a < b \in \mathbb{R} \cup \{\pm\infty\}$. Moreover, if we have two admissible pairs $(H_0, J_0), (H_1, J_1) \in \mathcal{N}_{reg}(M)$ such that $H_0(x) \leq H_1(x)$ for every $x \in \hat{M}$, then we can take a *monotone homotopy*, say (L, J), between them satisfying

$$L : \mathbb{R} \times S^1 \times \hat{M} \to \mathbb{R}, \quad L_s \in Ad(M),$$

$$\frac{\partial L}{\partial s} \geq 0, \quad L(s,t,x) = \begin{cases} H_0(t,x) & \text{if} \quad s \leq -s_0 \\ H_1(t,x) & \text{if} \quad s \geq s_0 \end{cases} \tag{5.1.4}$$

where $L_s(t,x) := L(s,t,x)$ and

$$J : \mathbb{R} \times S^1 \to \mathcal{J}^{SFT}_{\hat{\omega}}(\hat{M}), \quad J(s,t) = \begin{cases} J_0(t) & \text{if} \quad s \leq -s_0 \\ J_1(t) & \text{if} \quad s \geq s_0 \end{cases}$$

for some large $s_0 \in \mathbb{R}$. Using this pair (L, J), we can define moduli spaces

$$\begin{aligned} \mathcal{M}(x,y;L,J) \quad := \quad & \{u : \mathbb{R} \times S^1 \to \hat{M} | \partial_s u + J(s,t)(u)(\partial_t u - X_L(s,t,u)) = 0, \\ & \lim_{s \to -\infty} u(s,*) = x, \lim_{s \to +\infty} u(s,*) = y\}. \end{aligned}$$

for each $x \in \mathcal{P}_{H_0}, y \in \mathcal{P}_{H_1}$. Note that the condition (5.1.4) gives the compactness. For a generic (L, J), the moduli space $\mathcal{M}(x,y;L,J)$ is a smooth manifold of dimension $\mu_{CZ}(y) - \mu_{CZ}(y)$. Consider the degree 0 map

$$\phi^{(L,J)} : CF^{[a,b)}_k(H_0) \to CF^{[a,b)}_k(H_1), \quad \phi^{(L,J)}(x) := \sum_{\substack{y \in \mathcal{P}_{H_1}, \\ \mu_{CZ}(y)=k, \\ a \leq \mathcal{A}_H(y) < b}} \#_{\mathbb{Z}_2}\mathcal{M}(x,y;L,J)y.$$

This becomes a chain map between $CF_*(H_0)$ and $CF_*(H_1)$. Thus $\phi^{(L,J)}$ induces a natural map

$$\phi^{(L,J)}_{(H_0,H_1)} : FH^{[a,b)}_*(H_0) \to FH^{[a,b)}_*(H_1)$$

on the filtered Floer homology. In fact, one can prove that this map is independent of the choice of L by considering the homotopy of homotopies and hence we denote $\phi^{(L,J)}_{(H_0,H_1)}$ by $\phi_{(H_0,H_1)}$. The map $\phi_{(H_0,H_1)}$ is called the *monotone homomorphism between* H_0 *and* H_1. This defines a direct system

$$(\mathcal{N}_{reg}(M), \leq) \xrightarrow{FH^{[a,b)}} \mathcal{G}Ab$$

where $(\mathcal{N}_{reg}(M), \leq)$ is a directed set with the induced partial order from $Ad(M)$, namely $(H_0, J_0) \leq (H_1, J_1) \iff H_0(t,x) \leq H_1(t,x)$ for all $t \in S^1, x \in \hat{M}$ and $\mathcal{G}Ab$ is the category of graded abelian groups. We define the *symplectic homology*

$$SH_*^{[a,b)}(M,\omega) := \varinjlim FH_*^{[a,b)}(H) \tag{5.1.5}$$

of a Liouville domain $(M, \omega = d\lambda)$ *with filtration* $[a,b)$. Since direct limit preserves exactness, long exact sequence (5.1.3) induces a long exact sequence of symplectic homology

$$\cdots \to SH_*^{[a,b)}(M) \to SH_*^{[a,c)}(M) \to SH_*^{[b,c)}(M) \to SH_{*-1}^{[a,b)}(M) \to \cdots \tag{5.1.6}$$

for each $a < b < c \in \mathbb{R} \cup \{\pm\infty\}$. In particular, we obtain the following long exact sequence

$$\cdots \to SH_*^{<b}(M) \xrightarrow{i_M^b} SH_*(M) \xrightarrow{j_M^b} SH_*^{\geq b}(M) \to SH_{*-1}^{<b}(M) \xrightarrow{i_M^b} \cdots \tag{5.1.7}$$

by taking $a = -\infty, c = +\infty$ for each $b \in \mathbb{R}$. This will play an important role to define capacity in chapter 6. By definition of direct limit, we have the canonical map

$$\phi_H^{[a,b)} : FH_*^{[a,b)}(H) \to SH_*^{[a,b)}(M)$$

for each $(H,J) \in \mathcal{N}_{reg}(M)$ and canonical maps satisfy the following *universal property.*

$$\begin{tikzcd}
FH_*^{[a,b)}(H_i) \arrow[rr, "\phi^{[a,b)}_{(H_i,H_j)}"] \arrow[dr, "\phi^{[a,b)}_{H_i}"'] \arrow[ddr, "\psi_{H_i}"'] & & FH_*^{[a,b)}(H_j) \arrow[dl, "\phi^{[a,b)}_{H_j}"] \arrow[ddl, "\psi_{H_j}"] \\
& SH_*^{[a,b)}(M) \arrow[d, dashed, "\exists! \psi_M"'] & \\
& X_* &
\end{tikzcd}$$

Suppose that $(\hat{M}, \hat{\omega} = d\hat{\lambda})$ is an open exact symplectic manifold. We assume that there exist two Liouville domain $(M_1, \lambda_1) \subset (M_2, \lambda_2) \subset (\hat{M}, \hat{\lambda})$ such that can be identified $\hat{M}_1 = \hat{M}_2 = \hat{M}$. Then we have $Ad(M_2) \subset Ad(M_1)$ and so this induces a map

$$\phi^{[a,b)}_{M_1,M_2} : SH^{[a,b)}_*(M_2) \to SH^{[a,b)}_*(M_1) \tag{5.1.8}$$

for symplectic homology of M_1 and M_2. We call this map the *monotone morphism for* $M_1 \subset M_2$.

Example 5.1.7. Let $M_1 \subset M_2$ be compact star-shaped domains in $(\mathbb{R}^{2n}, \omega_{can} = d\lambda_{can})$. Then we can regard $\hat{M}_1 = \hat{M}_2 = \mathbb{R}^{2n}$ and therefore we have the monotone morphism

$$\phi^{[a,b)}_{M_1,M_2} : SH^{[a,b)}_*(M_2) \to SH^{[a,b)}_*(M_1)$$

on the symplectic homology. In [21], Floer and Hofer define monotone morphisms more generally for symplectic embeddings and in [22] Floer, Hofer and Wysocki use this morphism in order to study symplectic embeddings of ellipsoids in $\mathbb{R}^{2n}$ and to classify polydisks in $\mathbb{R}^{2n}$ symplectically. Moreover, they constructed a symplectic capacity for domains in $\mathbb{R}^{2n}$.

Example 5.1.8. Let $M_1 \subset M_2$ be fiberwise star-shaped domains in $(T^*N, \omega_{can} = d\lambda_{can})$. Then we have that $\hat{M}_1 = \hat{M}_2 = T^*N$. Thus we have the monotone morphism

$$\phi^{[a,b)}_{M_1,M_2} : SH^{[a,b)}_*(M_2) \to SH^{[a,b)}_*(M_1)$$

on the symplectic homology. Observing this monotone morphism, we will define a symplectic capacity for fiberwise star-shaped domains in a cotangent bundle in chapter 6.

We have defined the symplectic homology for a Liouville domain $(M, \omega = d\lambda)$. However, it is hard to see directly the computation of symplectic homology, the generators of the symplectic homology and so on. Because $Ad(M)$ is too big, we can consider a simpler set instead of $Ad(M)$. In $Ad(M)$, we allow only nondegenerate Hamiltonians and so we consider only the time-dependent Hamiltonian(A time-independent Hamiltonian is automatically a degenerate Hamiltonian due to S^1-action of each 1-periodic orbit).

Remark 5.1.9. If one uses the perturbation argument in [22], then it is possible to consider the time-independent Hamiltonian by requiring transversal nondegeneracy, that is, there is no eigenvalue 1 of linearized Hamiltonian flow for a 1-periodic orbit when it is restricted to the contact hyperplane. The Conley-Zehnder index defined above will be replaced by the transversal Conley-Zehnder index obtained by restricting the linearized flow to the contact plane. Moreover, we do not need to insist a smooth Hamiltonian if we use the remark about C^0-Hamiltonian in [51]. Hence we will assume that $Ad(M)$ contains transversely nondegenerate time-independent C^0-Hamiltonians which satisfy the original conditions as well.

Following the argument of Remark 5.1.9, we will use the following family of time-independent Hamiltonians

$$K_M^c(x) = \begin{cases} 0 & \text{if } x \in M \\ c(r-1) & \text{if } x = (r,p) \in [1,\infty) \times \partial M \end{cases}$$

on $\hat{M}$ for a Liouville domain $(M, \omega = d\lambda)$ and for $0 < c \notin Spec(\partial M, \lambda)$. See Figure 5.1.

Note that the family of functions $\{K_M^c\}_{c \in \mathbb{R}^+ \setminus Spec(\partial M, \lambda)}$ is cofinal in $Ad(M)$, that is, for any $H \in Ad(M)$ there exists $c \in \mathbb{R}^+$ such that $K_M^c \geq H$. This implies that

$$SH_*^{[a,b)}(M) = \varinjlim_{c} FH_*^{[a,b)}(K_M^c).$$

Let $H : \hat{M} \to \mathbb{R}$ be a time-independent Hamiltonian. We assume that H is C^2-small in M and $H(r,x) = h(r)$ on $(r,x) \in [1,\infty) \times \partial M$. Then 1-periodic orbits in M are all constant orbit at each of the critical points of H. We observe the symplectization cylinder part. Because we have

$$dH(r,x) = dh(r) = \frac{dh}{dr}(r)dr \implies X_H(r,x) = \frac{dh}{dr}(r)R_\lambda(r,x)$$

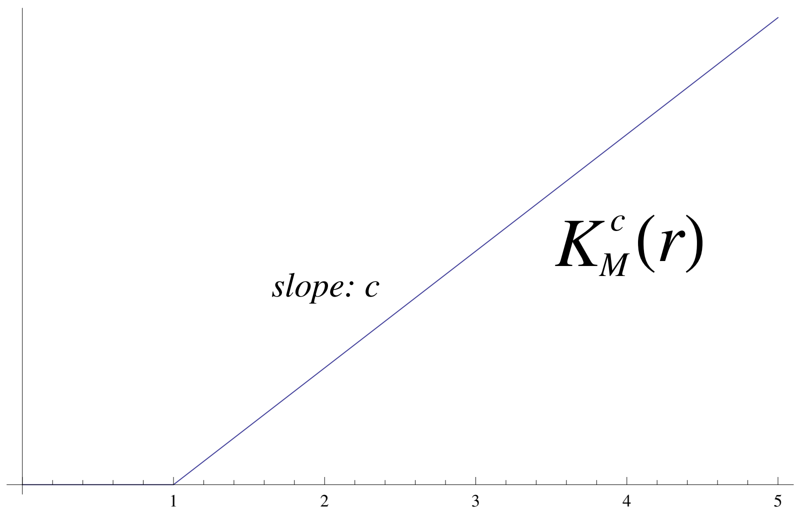

Figure 5.1: The graph of $K_M^c : \hat{M} \to \mathbb{R}$

for $(r, x) \in [1, \infty) \times \partial M$ where $R_\lambda(r, x) = (T_r)_*(R_\lambda(x))$ for the trivial map $T_r : \partial M \to \{r\} \times \partial M$. Let $x : S^1 \to [1, \infty) \times \partial M$ be a 1-periodic orbit of H. Then x lies on a level set, say $\{r\} \times \partial M$. Thus $\dot{x}(t) = \frac{dh}{dr}(r) R_\lambda(r, x(t))$ and so x is a copy of $\frac{dh}{dr}(r)$-periodic Reeb orbit. Moreover, we have the action value

$$\begin{aligned}
\mathcal{A}_H(x) &= \int_{S^1} x^* \hat{\lambda} - \int_0^1 H(x(t)) dt \\
&= \int_0^1 \hat{\lambda}(\frac{dh}{dr}(r) R_\lambda(r, x)) - \int_0^1 h(r) dt \\
&= r \frac{dh}{dr}(r) - h(r)
\end{aligned}$$

of x in terms of r, h. Let us discuss 1-periodic orbits of the Hamiltonian K_M^c. We assume that $c \notin Spec(\partial M, \lambda)$ and denote $K_M^c(x, r) = k_M^c(r)$ on the cylinder. In the function k_M^c, every slope between 0 and c appears exactly once near $r = 1$. This implies that the 1-periodic orbits of K_M^c have one-to-one correspondence with the periodic Reeb orbits of period $T \in (0, c)$ in $(\partial M, \lambda)$. Moreover, the action value of a 1-periodic orbit is given by its corresponding Reeb period T.

We shall introduce the symplectic homology for our previous examples. Our first example is the star-shaped domain in $\mathbb{R}^{2n}$. It is the simplest example for symplectic homology. In particular, computations of symplectic homologies with any action filtration for ellipsoids and polydisks was done in [22]. We borrow the result.

Example 5.1.10 (Symplectic homology of ellipsoid)**.** Let $r = (r_1, r_2, \cdots, r_n)$ be an n-tuple of positive real numbers such that $r_1 \le r_2 \le \cdots \le r_n$. We define an open ellipsoid

$$E(r) := \left\{ z \in \mathbb{C}^n | \sum_{k=1}^{n} \left| \frac{z_k}{r_k} \right|^2 < 1 \right\}$$

in $\mathbb{C}^n$. We define the set

$$\sigma(r) := \left\{ k\pi r_j^2 | k \in \mathbb{N}, j \in \{1, 2, \cdots, n\} \right\} = \{d_1 \le d_2 \le \cdots\}$$

that allows repeated elements according to the multiplicity. For every $d \in \mathbb{R} \cup \{+\infty\}$, we define a chain complex

$$\begin{aligned} C^d(r) &= 0 \quad \text{for} \quad d \le 0 \\ C^d(r) &= (\mathbb{Z}_2, n) \quad \text{for} \quad 0 < d \le d_1 \\ C^d(r) &= (\mathbb{Z}_2, n) \oplus (\mathbb{Z}_2, n+1) \oplus \cdots \oplus (\mathbb{Z}_2, n + 2m(d, r)) \quad \text{for} \quad d_1 \le d < +\infty \\ C^{+\infty}(r) &= \bigoplus_{l=0}^{+\infty} (\mathbb{Z}_2, n + l) \end{aligned}$$

where the right component denotes the grade and $m(d, r) := \sup\{l | d_l < d\}$. We also define its quotient

$$C^{[a,b)}(r) := C^b(r)/C^a(r)$$

The boundary map

$$\cdots \xrightarrow{id} (\mathbb{Z}_2, n + 2m) \xrightarrow{0} (\mathbb{Z}_2, n + 2m - 1) \xrightarrow{id} \cdots$$

$$\xrightarrow{id} (\mathbb{Z}_2, n + 2) \xrightarrow{0} (\mathbb{Z}_2, n + 1) \xrightarrow{id} (\mathbb{Z}_2, n) \xrightarrow{0} 0$$

of infinite chain complex gives the boundary map for each $C^d(r)$ or $C^{[a,b)}$ by restriction. The following result was proven in [22].

$$SH_*^{[a,b)}(E(r)) = H_*(C^{[a,b)}(r), \partial^{[a,b)}(r)).$$

In particular, we have $SH_*(E(r)) = 0$.

In [22], they answered many embedding problems between two ellipsoids from the information of periodic Reeb orbits because we know every periodic Reeb orbits on $\partial E(r)$. In this thesis, we will work in the opposite way. Namely, we will obtain information of periodic Reeb orbit from the embedding relations.

We shall see the symplectic homology of our another example. This computation was done in [1], [49] and [52] independently. We will follow the proof of Abbondandolo-Schwarz.

Example 5.1.11 (Symplectic homology of cotangent bundle)**.** Assume that N is a closed orientable manifold. Let M be a fiberwise star-shaped domain in $(T^*N, \omega_{can} = d\lambda_{can})$. Then we have the following result.

Theorem for Floer homology of a cotangent bundle (Abbondandolo-Schwarz [1], Salamon-Weber [49], Viterbo [52])**.** The symplectic homology $SH_*(M)$ is isomorphic to the homology $H_*(\Lambda N)$ of the free loop space of N.

We will give a sketch of proof for this result. In [1], they regarded a symplectic homology as a Floer homology on the cotangent bundle and defined special conditions for Hamiltonian $H : S^1 \times T^*N \to \mathbb{R}$ as follows.

- (**H1**): $dH(t,q,p)[Y_{can}] - H(t,q,p) \geq c_0|p|^2 - h_1$ for some constants $c_0 > 0, c_1 \geq 0$.
- (**H2**): $|\nabla_q H(t,q,p) \leq c_2(1+|p|^2), |\nabla_p H(t,q,p) \leq c_2(1+|p|^2)$ for some constant $c_2 \geq 0$.

Let $Qd(T^*N)$ be the set of Hamiltonians which satisfy the above conditions (**H1**) and (**H2**). A difficulty of this extension of function is the compactness of moduli spaces. Instead of using the maximum principle, they observe directly the Cauchy-Riemann operator and they get an L^∞ estimate. The conditions (**H1**) and (**H2**) allow a Hamiltonian H to have a Lagrangian L satisfying

- (**L1**): $\nabla_{vv}L(t,q,v) \geq d_0 I$ for some constant $d_0 > 0$.
- (**L2**): $|\nabla_{qq}L(t,q,v)| \leq d_1(1+|v|^2), \nabla_{qv}L(t,q,v)| \leq d_1(1+|v|)$ and $|\nabla_{vv}L(t,q,v)| \leq d_1$ for some constant $d_1 \geq 0$.

by the Legendre transformation. Using this Lagrangian L, one can consider the Lagrangian energy functional

$$\mathcal{E}_L(x) = \int_0^1 L(t,x(t),\dot{x}(t))dt$$

on the free loop space $x \in \Lambda N := W^{1,2}(S^1,N)$ of N. They developed the Morse homology on ΛN using $\mathcal{E}_L$ and defined an isomorphism

$$\Theta : (CM_*(\mathcal{E}_L), \partial_*(\mathcal{E}_L, g)) \to (CF_*(H), \partial_*(H,J))$$

on the chain levels where g is a Morse-Smale Riemannian metric on ΛN. This proves the isomorphism between the symplectic homology $SH_*(M)$ and the Morse homology $H_*(\Lambda N)$. This will play an important role to define symplectic capacity using a min-max argument.

We shall finish this section with one more example. It is a particular case of Example 5.1.11. We will use this to apply the symplectic capacity defined in this thesis to Hill's lunar problem.

Example 5.1.12 (Symplectic homology of T^*S^2)**.** From the Theorem for Floer homology of a cotangent bundle, we have that

$$SH_*(M) \cong H_*(\Lambda S^2)$$

for any fiberwise star-shaped domain $M \in T^*S^2$. We know the homology of ΛS^2 from the result of string topology in [16] including general computations for loop homologies of spheres and projective spaces.

$$H_*(\Lambda S^2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } \quad * = 0, 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } \quad * \geq 2 \end{cases}$$

5.2 Wrapped Floer homology of Liouville domain

Wrapped Floer homology has several interpretations. One can understand wrapped Floer homology as Lagrangian Floer homology for open symplectic manifold or an open string version of symplectic homology. In the thesis, we will emphasize the action filtration and try to keep the view of the analogue of symplectic homology case. In this thesis, we will use the wrapped Floer homology for cotangent bundles. However, except the Maslov indices argument, the derivation of wrapped Floer homology on Liouville domains is similar to that on cotangent bundles. Thus we will discuss more generally on Liouville domains. We will assume that Maslov indices are well-defined. One can find more general conditions which impose the well-definedness of Maslov indices in [3] and [4]. We have to be able to assign Maslov indices of Hamiltonian chords on cotangent bundles for our application. For Maslov indices of Hamiltonian chords on cotangent bundles, we will use the definition in [2]. We will leave the index discussion on cotangent bundles in Appendix A.2.

Let $(M, \omega = d\lambda)$ be a Liouville domain with Liouville vector field Y. We are interested in the Lagrangian $\mathcal{L}$ satisfying the following conditions. These are strong conditions.

- $(\mathcal{L}\mathbf{1})$: $Y_q \in T_q\mathcal{L}$ for all $q \in \mathcal{L}$.
- $(\mathcal{L}\mathbf{2})$: $\mathcal{L}$ is transverse to ∂M.
- $(\mathcal{L}\mathbf{3})$: $\mathcal{L}|_\lambda = 0$.

One can find relaxed conditions in [3] and [4] as well. A pair of Lagrangian $(\mathcal{L}_0, \mathcal{L}_1)$ is called *admissible* if each satisfies $(\mathcal{L}\mathbf{1})$, $(\mathcal{L}\mathbf{2})$, $(\mathcal{L}\mathbf{3})$ and the pair of Legendrian $(\partial\mathcal{L}_0, \partial\mathcal{L}_1)$ is nondegenerate. Note that $\mathcal{L}_0$ and $\mathcal{L}_1$ are not necessarily different. We can think of the completion of $\mathcal{L}_i$ as well by our assumptions. That is, $\mathcal{L}_i$ extends to the Lagrangian submanifold

$$\hat{\mathcal{L}}_i = \mathcal{L}_i \cup_{\{1\}\times\partial\mathcal{L}_i} [1, \infty) \times \partial\mathcal{L}_i$$

of the completion $(\hat{M}, \hat{\omega})$ for each $i = 0, 1$.

Example 5.2.1 (Star-shaped domain in $\mathbb{R}^{2n}$ with Lagrangian subspace)**.** We have seen that $(B_1^{2n}(0), \omega_0 = d\lambda_0)$ is a Liouville domain. If we take $\mathcal{L} = \{p = 0\} \cap B_1^{2n}(0)$, then this pair satisfies ($\mathcal{L}$**1**), ($\mathcal{L}$**2**) and ($\mathcal{L}$**3**). More generally $(D, \omega_0 = d\lambda_0)$ is a Liouville domain provided D is a star-shaped domain in $\mathbb{R}^{2n}$. If Λ is a Lagrangian subspace, then $\Lambda \cap D$ satisfies ($\mathcal{L}$**1**), ($\mathcal{L}$**2**) and ($\mathcal{L}$**3**).

Example 5.2.2 (Fiberwise star-shaped domain in T^*N with conormal bundle)**.** Let (N, g) be a closed Riemannian manifold and Q is a submanifold of N. A fiberwise star-shaped domain $(M, \omega_{can} = d\lambda_{can})$ with canonical 1-form in T^*N defines a Liouville domain by Proposition 4.3.2. If $\mathcal{L}$ is the intersection $M \cap \nu^*Q$, then $\mathcal{L}$ satisfies the conditions ($\mathcal{L}$**1**), ($\mathcal{L}$**2**) and ($\mathcal{L}$**3**).

Example 5.2.3 (Real Liouville domain)**.** Let $(M, \lambda, \mathcal{R})$ be a real Liouville domain. Then we have a Lagrangian submanifold $\mathcal{L} = \mathrm{Fix}(\mathcal{R})$ of M if $\mathrm{Fix}(\mathcal{R})$ is nonempty. If $\mathcal{L}$ satisfies the conditions ($\mathcal{L}$**1**), ($\mathcal{L}$**2**) and ($\mathcal{L}$**3**), then this pair provides an example. As a concrete example, we consider $(T^*S^2, \omega_{can} = d\lambda)$ and we identify

$$T^*S^2 = \{((x_1, x_2, x_3), (y_1, y_2, y_3)) \in T^*\mathbb{R}^3 | |x| = 1, x \cdot y = 0\}.$$

Define the following map

$$\mathcal{R}_1 : T^*S^2 \to T^*S^2, \quad ((x_1, x_2, x_3), (y_1, y_2, y_3)) \mapsto ((-x_1, x_2, x_3), (y_1, -y_2, -y_3)).$$

Then it is easy to see that $\mathcal{R}_1^2 = \mathrm{id}$ and $\mathcal{R}_1^*\lambda_{can} = -\lambda_{can}$. Therefore, if we take a fiberwise star-shaped domain M in T^*S^2 which is invariant under $\mathcal{R}_1$, then $(M, \lambda_{can}, \mathcal{R}_1)$ is a real Liouville domain. We have that

$$\mathrm{Fix}(\mathcal{R}_1) = \{((0, x_2, x_3), (y_1, 0, 0)) \in T^*S^2\}$$

and this is the conormal bundle $\nu^* Q_1$ of the equator $Q_1 = \{(0, x_2, x_3) \in S^2\}$. We can also define $\mathcal{R}_2$ and $\mathcal{R}_3$. If we define $\mathcal{L}_i := \mathrm{Fix}(\mathcal{R}_i) \cap M$, then

$$((M, d\lambda_{can}), (\mathcal{L}_i, \mathcal{L}_j)), \quad i, j \in \{1, 2, 3\}$$

becomes the pair satisfying the conditions ($\boldsymbol{\mathcal{L}1}$), ($\boldsymbol{\mathcal{L}2}$) and ($\boldsymbol{\mathcal{L}3}$) for any $\mathcal{R}_i, \mathcal{R}_j$-invariant fiberwise star-shaped domain M in T^*S^2.

Wrapped Floer homology $WFH_*(\mathcal{L}_0, \mathcal{L}_1)$ can be obtained by taking a limit on a family of Lagrangian Floer homology for the pair $(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1)$ by perturbing $\hat{\mathcal{L}}_0$ with carefully chosen family of Hamiltonians. First, we will define the Lagrangian Floer homology $LFH(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1, H)$ with a Hamiltonian H and later we will specify the type of Hamiltonian that we use for the wrapped Floer homology.

Let $(\hat{M}, \hat{\omega} = d\hat{\lambda})$ be the completion of a Liouville domain (M, ω). We choose a time-dependent Hamiltonian $H : [0,1] \times \hat{M} \to \mathbb{R}$. We define $H_t(x) = H(t, x)$ for notational convenience. We define the *action functional*

$$\mathcal{A}_H(x) = \int_{[0,1]} x^*\hat{\lambda} - \int_0^1 H(t, x(t))dt$$

associated to H *on the path space* $\mathcal{P}_{\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1}\hat{M} := \{x \in C^\infty([0,1], \hat{M}) | x(0) \in \hat{\mathcal{L}}_0, x(1) \in \hat{\mathcal{L}}_1\}$ *of* $\hat{M}$. Following the computations in section 5.1, we can derive the differential of $\mathcal{A}_H$

$$d\mathcal{A}_H(x)(\hat{v}) = \int_0^1 \hat{\omega}(\hat{v}(t), \dot{x}(t) - X_H^t(x(t)))dt + \lambda(x(1))(\hat{v(1)}) - \lambda(x(0))(\hat{v(0)})$$

for a tangent vector $\hat{v} \in T_x\mathcal{P}_{\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1}\hat{M}$ at $x \in \mathcal{P}_{\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1}\hat{M}$ where $\dot{} = \frac{d}{dt}$. Here, we interpret the tangent vector $\hat{v} \in T_x\mathcal{P}_{\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1}\hat{M}$ as a section of pull-back bundle $x^*T\hat{M}$ satisfying $\hat{v}(0) \in T_{x(0)}\hat{\mathcal{L}}_0, \hat{v}(1) \in T_{x(1)}\hat{\mathcal{L}}_1$. The boundary terms vanish by the condition ($\mathcal{L}\mathbf{3}$). Thus we have that

$$Crit(\mathcal{A}_H) = \{x \in \mathcal{P}_{\hat{\mathcal{L}}_0,\hat{\mathcal{L}}_1}\hat{M} | \dot{x}(t) = X^t_H(x(t))\}$$

and we will denote by $\mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$ the *set of all Hamiltonian 1-chord of H from $\hat{\mathcal{L}}_0$ to $\hat{\mathcal{L}}_1$.*

Definition 5.2.4. A Hamiltonian 1-chord $x \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$ is called *nondegenerate* if the image $d\phi^1_H(x(0))(T_{x(0)}\hat{\mathcal{L}}_0)$ of the tangent space of $\hat{\mathcal{L}}_0$ along the linearized Hamiltonian flow transverses to $T_{x(1)}\hat{\mathcal{L}}_1$. We call a Hamiltonian $H \in C^\infty([0,1] \times \hat{M})$ *nondegenerate* if every $x \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$ is nondegenerate.

Nondegeneracy is a generic condition and we will assume our Hamiltonian H is nondegenerate. We choose a family of SFT-like $\hat{\omega}$-compatible almost complex structure $J = \{J_t\}_{t\in[0,1]}$. Following the derivation in section 5.1, we have the perturbed Cauchy-Riemann equation

$$u : \mathbb{R} \times [0,1] \to \hat{M}, \quad \partial_s u + J_t(u)(\partial_t u - X^t_H(u)) = 0 \tag{5.2.1}$$

satisfying the boundary conditions $u(s,0) \in \hat{\mathcal{L}}_0$ and $u(s,1) \in \hat{\mathcal{L}}_1$ for all $s \in \mathbb{R}$. This partial differential equation is called *Floer equation on the infinite strip.*

The boundary map is given by counting the solution of Floer equation having fixed asymptotic chords. Define the *moduli space of the gradient flow lines from x^- to x^+ for $x^\pm \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$.*

$$\widehat{\mathcal{M}}(x^-, x^+; H, J) = \{u : \mathbb{R} \times [0,1] \to \hat{M} | (5.2.1), \lim_{s\to\pm\infty} u(s,t) = x^\pm(t),$$
$$u(s,0) \in \hat{\mathcal{L}}_0, u(s,1) \in \hat{\mathcal{L}}_1\}.$$

We define the *unparametrized moduli space* $\mathcal{M}(x^-, x^+; H, J)$ by quotient the free $\mathbb{R}$-action on $\widehat{\mathcal{M}}(x^-, x^+; H, J)$. We can assume that all elements in $\mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$ and the gradient trajectories between them are contained in the compact subset of $\hat{M}$. For a generic $J \in C^\infty([0,1], \mathcal{J}^{SFT}_{\hat{\omega}}(\hat{M}))$, the moduli space $\mathcal{M}(x^-, x^+; H, J)$ is a smooth manifold of dimension $\mu(x^+) - \mu(x^-) - 1$ for each $x^-, x^+ \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$. We define the following *filtered Floer chain* *group for* $\mathcal{L}_0, \mathcal{L}_1$ *and* H

$$CF_k^{<a}(\mathcal{L}_0, \mathcal{L}_1; H) := \mathbb{Z}_2 \langle x \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H) | \mu(x) = k, \mathcal{A}_H(x) < a \rangle$$

as the $\mathbb{Z}_2$-module generated by Hamiltonian chords of index k and action less than a for $k \in \mathbb{Z}$ and $a \in \mathbb{R} \cup \{\pm\infty\}$. We abbreviate $CF_k^{<+\infty}(\mathcal{L}_0, \mathcal{L}_1; H) = CF_k(\mathcal{L}_0, \mathcal{L}_1; H)$. Define the *filtered chain complex*

$$CF_k^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H) := CF_k^{<b} / CF_k^{<a}$$

of action filtration $a < b \in \mathbb{R} \cup \{\pm\infty\}$ and define a *boundary map*

$$\partial^{[a,b)} : CF_k^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H) \to CF_{k-1}^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H),$$

$$\partial^{[a,b)}(x) := \sum_{\substack{y \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H), \\ \mu(y)=k-1, \\ a \leq \mathcal{A}_H(y) < b}} \#_{\mathbb{Z}_2} \mathcal{M}(y, x; J, H) y$$

If we have compactness for the moduli spaces, then $\partial^{[a,b)}$ is well defined and it satisfies $\partial^{[a,b)} \circ \partial^{[a,b)} = 0$. Under the compactness assumption, we can define the *filtered Lagrangian Floer homology groups*

$$LFH_*^{[a,b)}(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1; H, J) = \ker \partial^{[a,b)} / im\ \partial^{[a,b)}$$

associated with Hamiltonian H *for* $a < b \in \mathbb{R} \cup \{\pm\infty\}$. A priori, the boundary map $\partial^{[a,b)}$ depends on the choice of J. However we can show that the Lagrangian Floer homology does not depend on the choice of the family of the almost complex structures J using the standard homotopy argument. Thus, we will hide J from now for the notational convenience. From a short exact sequence of chain complexes

$$0 \to CF_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H) \to CF_*^{[a,c)}(\mathcal{L}_0, \mathcal{L}_1; H) \to CF_*^{[b,c)}(\mathcal{L}_0, \mathcal{L}_1; H) \to 0,$$

we have a long exact sequence of the filtered Lagrangian Floer homology groups

$$\cdots \to LFH_*^{[a,b)}(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1; H) \to LFH_*^{[a,c)}(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1; H) \to LFH_*^{[b,c)}(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1; H)$$

$$\to LFH_{*-1}^{[a,b)}(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1; H) \to \cdots \tag{5.2.2}$$

Now we have to specify the Hamiltonians which give the desired properties for compactness.

Definition 5.2.5. We call a smooth Hamiltonian $H : [0,1] \times \hat{M} \to \mathbb{R}$ *admissible for* $(M, \mathcal{L}_0, \mathcal{L}_1)$ if it satisfies the following conditions

- H is nondegenerate.
- $H|_{[0,1]\times M} \leq 0$
- $\lim_{r\to\infty} H(\cdot, r, x) = ar + b$ on symplectic cylinder $(r, x) \in [1, +\infty) \times \partial M$ for some $a, b \in \mathbb{R}$ such that $0 < a \notin Spec(\partial M, \lambda; \partial\mathcal{L}_0, \partial\mathcal{L}_1)$.

We denote by $Ad(\mathcal{L}_0, \mathcal{L}_1, M)$ the *set of all admissible Hamiltonian on* $\hat{M}$.

For an admissible Hamiltonian $H \in Ad(\mathcal{L}_0, \mathcal{L}_1, M)$, there is a $[0,1]$-family of SFT-like $\hat{\omega}$-compatible almost complex structure J such that the moduli space $\mathcal{M}(x^-, x^+; H, J)$ is a smooth manifold for each $x^-, x^+ \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H)$. Moreover, in fact, the set of all such $[0,1]$-family of SFT-like $\hat{\omega}$-compatible almost complex structure forms a Baire set in $C^\infty([0,1], \mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M}))$. Such a pair $(H, J) \in Ad(\mathcal{L}_0, \mathcal{L}_1, M) \times C^\infty([0,1], \mathcal{J}_{\hat{\omega}}^{SFT}(\hat{M}))$ is called an *admissible pair*. We denote by $\mathcal{N}_{reg}(\mathcal{L}_0, \mathcal{L}_1, M)$ the *set of all admissible pairs*. For an admissible pair $(H, J) \in \mathcal{N}_{reg}(\mathcal{L}_0, \mathcal{L}_1, M)$, we can define the filtered Lagrangian Floer homology

$$LFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H, J)$$

for $a < b \in \mathbb{R} \cup \{\pm\infty\}$, because admissible Hamiltonians guarantee the compactness of the moduli spaces and its compatible almost complex structure guarantees smoothness of the moduli space as we desired. Moreover, if we have two admissible pairs $(H_0, J_0), (H_1, J_1) \in \mathcal{N}_{reg}(\mathcal{L}_0, \mathcal{L}_1, M)$ such that

$H_0(x) \leq H_1(x)$ for every $x \in \hat{M}$, then we can take a *monotone homotopy*, say (L, J), between them satisfying

$$L : \mathbb{R} \times [0,1] \times \hat{M} \to \mathbb{R}, \quad L_s \in Ad(\mathcal{L}_0, \mathcal{L}_1, M), \quad \frac{\partial L}{\partial s} \geq 0,$$
$$L(s,t,x) = \begin{cases} H_0(t,x) & \text{if} \quad s \leq -s_0 \\ H_1(t,x) & \text{if} \quad s \geq s_0 \end{cases}$$

where $L_s(t,x) := L(s,t,x)$ and

$$J : \mathbb{R} \times [0,1] \to \mathcal{J}^{SFT}_{\hat{\omega}}(\hat{M}), \quad J(s,t) = \begin{cases} J_0(t) & \text{if} \quad s \leq -s_0 \\ J_1(t) & \text{if} \quad s \geq s_0 \end{cases}$$

for some large $s_0 \in \mathbb{R}$. Using this pair (L, J), we can define moduli spaces

$$\begin{aligned} \mathcal{M}(x,y;L,J) \quad := \quad & \{u : \mathbb{R} \times [0,1] \to \hat{M} | \partial_s u + J(s,t)(u)(\partial_t u - X_L(s,t,u)) = 0, \\ & \lim_{s \to -\infty} u(s,*) = x, \lim_{s \to +\infty} u(s,*) = y\}. \end{aligned}$$

for each $x \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H_0), y \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H_1)$. For a generic (L, J), the moduli space $\mathcal{M}(x,y;L,J)$ is a smooth manifold of dimension $\mu(y) - \mu(y)$. If we consider the degree 0 map

$$\phi^{(L,J)} : CF_k^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H_0) \to CF_k^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H_1),$$

$$\phi^{(L,J)}(x) := \sum_{\substack{y \in \mathcal{P}_{\mathcal{L}_0,\mathcal{L}_1}(H_1), \\ \mu(y)=k, \\ a \leq \mathcal{A}_H(y) < b}} \#_{\mathbb{Z}_2} \mathcal{M}(x,y;L,J) y.$$

Then this is a chain map between $CF_*(\mathcal{L}_0, \mathcal{L}_1; H_0)$ and $CF_*(\mathcal{L}_0, \mathcal{L}_1; H_1)$. Thus $\phi^{(L,J)}$ induces a natural map

$$\phi^{(L,J)}_{(H_0,H_1)} : LFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H_0) \to LFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H_1)$$

on the filtered Floer homology. We denote $\phi^{(L,J)}_{(H_0,H_1)}$ by $\phi_{(H_0,H_1)}$ and call it the *monotone homomorphism between* H_0 *and* H_1. We have a direct system

$$(\mathcal{N}_{reg}(\mathcal{L}_0, \mathcal{L}_1, M), \leq) \xrightarrow{FH^{[a,b)}} \mathcal{G}Ab$$

where $(\mathcal{N}_{reg}(\mathcal{L}_0, \mathcal{L}_1, M), \leq)$ is a directed set with the induced partial order from $Ad(\mathcal{L}_0, \mathcal{L}_1, M)$, namely $(H_0, J_0) \leq (H_1, J_1) \iff H_0(t,x) \leq H_1(t,x)$ for all $t \in [0,1], x \in \hat{M}$ and $\mathcal{G}Ab$ is the category of graded abelian groups. We define the wrapped Floer homology

$$WFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1, M, \omega) := \varinjlim LFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H)$$

of the pair of admissible Lagrangians $(\mathcal{L}_0, \mathcal{L}_1)$ in a Liouville domain $(M, \omega = d\lambda)$ with filtration $[a,b)$. It will be denoted by $WFH(\mathcal{L}_0, \mathcal{L}_1)$ when the Liouville domain $(M, \omega = d\lambda)$ is clear in the context. We denote by $WFH_*(\mathcal{L}; M)$ when $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}$. From the naturality of direct limit and (5.2.2), we have a long exact sequence of wrapped Floer homology

$$\begin{aligned} \cdots \to WFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1) \to WFH_*^{[a,c)}(\mathcal{L}_0, \mathcal{L}_1) \to WFH_*^{[b,c)}(\mathcal{L}_0, \mathcal{L}_1) \\ \to WFH_{*-1}^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1) \to \cdots \end{aligned} \tag{5.2.3}$$

for each $a < b < c \in \mathbb{R} \cup \{\pm\infty\}$. In particular, we obtain the following long exact sequence

$$\begin{aligned} \cdots \to WFH_*^{<b}(\mathcal{L}_0, \mathcal{L}_1) \xrightarrow{i_M^b} WFH_*(\mathcal{L}_0, \mathcal{L}_1) \xrightarrow{j_M^b} WFH_*^{\geq b}(\mathcal{L}_0, \mathcal{L}_1) \\ \to WFH_{*-1}^{<b}(\mathcal{L}_0, \mathcal{L}_1) \xrightarrow{i_M^b} \cdots \end{aligned} \tag{5.2.4}$$

by taking $a = -\infty, c = +\infty$ for each $b \in \mathbb{R}$. This will play an important role to define capacity in chapter 6. By definition of direct limit, we have the canonical map

$$\phi_H^{[a,b)} : LFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; H) \to WFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1)$$

for each $(H, J) \in \mathcal{N}_{reg}(M)$ and these canonical maps satisfy the following *universal property.*

$$\begin{tikzcd}
LFH_*^{[a,b]}(\mathcal{L}_0,\mathcal{L}_1;H_i) \arrow[rr, "\phi^{[a,b]}_{(H_i,H_j)}"] \arrow[dr, "\phi^{[a,b]}_{H_i}"'] \arrow[ddr, "\psi_{H_i}"'] & & LFH_*^{[a,b]}(\mathcal{L}_0,\mathcal{L}_1;H_j) \arrow[dl, "\phi^{[a,b]}_{H_j}"] \arrow[ddl, "\psi_{H_j}"] \\
& WFH_*^{[a,b]}(\mathcal{L}_0,\mathcal{L}_1) \arrow[d, dashed, "\exists!\psi_M"'] & \\
& X_* &
\end{tikzcd}$$

Suppose that $(\hat{M},\hat{\omega}=d\hat{\lambda})$ is an open exact symplectic manifold. We assume that there exist two Liouville domain $(M_1,\lambda_1)\subset(M_2,\lambda_2)\subset(\hat{M},\hat{\lambda})$ such that we can identify $\hat{M}_1=\hat{M}_2=\hat{M}$. Moreover we have two pairs of admissible Lagrangians $(\mathcal{L}_0^1,\mathcal{L}_1^1)$ and $(\mathcal{L}_0^2,\mathcal{L}_1^2)$ of M_1 and M_2, respectively, such that $\hat{\mathcal{L}}_0^1=\hat{\mathcal{L}}_0^2$ and $\hat{\mathcal{L}}_1^1=\hat{\mathcal{L}}_1^2$. Then we have $Ad(\mathcal{L}_0^2,\mathcal{L}_1^2,M_2)\subset Ad(\mathcal{L}_0^1,\mathcal{L}_1^1,M_1)$ and so this induces a map

$$\phi^{[a,b]}_{M_1,M_2}:WFH_*^{[a,b]}(\mathcal{L}_0^2,\mathcal{L}_1^2,M_2)\to WFH_*^{[a,b]}(\mathcal{L}_0^1,\mathcal{L}_1^1,M_1)$$

on wrapped Floer homologies of M_1 and M_2. We call this map the *monotone morphism.*

We have defined the wrapped Floer homology for a pair of admissible Lagrangians $(\mathcal{L}_0,\mathcal{L}_1)$ in a Liouville domain $(M,\omega=d\lambda)$. Following the argument in section 5.1, we consider the Hamiltonian

$$K_M^c(x)=\begin{cases}0 & \text{if } x\in M\\ c(r-1) & \text{if } x=(r,p)\in[1,\infty)\times\partial M\end{cases}$$

on $\hat{M}$ for a Liouville domain $(M,\omega=d\lambda)$ and for $0<c\notin Spec(\partial M,\lambda;\partial\mathcal{L}_0,\partial\mathcal{L}_1)$. We note that the family $\{K_M^c\}_{c\in\mathbb{R}^+\setminus Spec(\partial M,\lambda;\partial\mathcal{L}_0,\partial\mathcal{L}_1)}$ of functions is cofinal in

$Ad(\mathcal{L}_0, \mathcal{L}_1, M)$. This implies that

$$WFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1) = \varinjlim_{c} LFH_*^{[a,b)}(\mathcal{L}_0, \mathcal{L}_1; K_M^c).$$

Let $H : \hat{M} \to \mathbb{R}$ be a time-independent Hamiltonian. We assume that H takes value 0 in M and $H(r,x) = h(r)$ on $(r,x) \in [1,\infty) \times \partial M$. Then Hamiltonian 1-chord from $\mathcal{L}_0$ to $\mathcal{L}_1$ in M are all constant chord on the intersection point of $\mathcal{L}_0$ and $\mathcal{L}_1$. We observe the symplectization cylinder part. Because we have

$$dH(r,x) = dh(r) = \frac{dh}{dr}(r)dr \implies X_H(r,x) = \frac{dh}{dr}(r)R_\lambda(r,x)$$

for $(r,x) \in [1,\infty) \times \partial M$ where $R_\lambda(r,x) = (T_r)_*(R_\lambda(x))$ for the trivial map $T_r : \partial M \to \{r\} \times \partial M$. Let $x : [0,1] \to [1,\infty) \times \partial M$ be a Hamiltonian 1-chord of H. Then x lies on a level set, say $\{r\} \times \partial M$. Thus $\dot{x}(t) = \frac{dh}{dr}(r)R_\lambda(r, x(t))$ and so x is a copy of Reeb $\frac{dh}{dr}(r)$-chord, that is, one can identify the Hamiltonian 1-chord from $\hat{\mathcal{L}}_0$ to $\hat{\mathcal{L}}_1$ with the Reeb chord $v : [0, \frac{dh}{dr}(r)] \to \partial M$ from $\partial \mathcal{L}_0$ to $\partial \mathcal{L}_1$. Moreover, we have the action value

$$\mathcal{A}_H(x) = r\frac{dh}{dr}(r) - h(r)$$

of x in terms of r, h. Let us discuss 1-chord of the Hamiltonian K_M^c. We assume that $c \notin Spec(\partial M, \lambda; \partial \mathcal{L}_0, \partial \mathcal{L}_1)$ and denote $K_M^c(x,r) = k_M^c(r)$ on the cylinder. In the function k_M^c, we can think that every slope between 0 and c appears exactly once and arbitrarily close to $\{1\} \times \partial M$. This implies that the Hamiltonian 1-chords of K_M^c have one-to-one correspondence with the Reeb chord of length $T \in (0,c)$ in $(\partial M, \lambda)$. Moreover, the action value of a Hamiltonian 1-chord is given by the length T of corresponding Reeb chord.

Example 5.2.6 (Wrapped Floer homology for star-shaped domains in $\mathbb{R}^{2n}$ with Lagrangian $\{p = 0\}$)**.** We consider a star-shaped domain D, with respect to the origin, in the standard symplectic space $(\mathbb{C}^n, \omega_{can} = d\lambda_{can})$. Then $(D, \omega_{can} = d\lambda_{can})$ is a Liouville domain. In addition, we consider the conjugation map

$$\mathcal{R} : \mathbb{C}^n \to \mathbb{C}^n, \quad \mathcal{R}(z_1, \cdots, z_n) = (\bar{z}_1, \cdots, \bar{z}_n)$$

on $\mathbb{C}^n$. This is an anti-symplectic involution on $\mathbb{C}^n$. For $z_j = q_j + ip_j$, the fixed point set $\mathrm{Fix}(\mathcal{R})$ is the Lagrangian $\{p = 0\}$. If D is invariant under this $\mathcal{R}$, then the triple $(D, \lambda_{can}, \mathcal{R})$ becomes a real Liouville domain. The wrapped Floer homology

$$WFH_*(\mathcal{L}; D) = 0$$

was computed in [29].

We shall see the wrapped Floer homology for our another example. This computation was done in [2].

Example 5.2.7 (Wrapped Floer homology for fiberwise star-shaped domains in T^*N with conormal Lagrangians)**.** Let M be a fiberwise star-shaped domain in $(T^*N, \omega_{can} = d\lambda_{can})$. Let Q_0 and Q_1 be submanifolds of N. We borrow the following Theorem in [2]. In fact, they proved the Theorem for more general situation: non-local boundary condition. We need the following partial result.

Theorem for Floer homology with conormal boundary conditions (Abbondandolo-Portaluri-Schwarz, [2])**.** There is a isomorphism between the wrapped Floer homology $WFH_*(\nu^*Q_0, \nu^*Q_1; M)$ and the singular homology $H_*(\mathcal{P}_{Q_0,Q_1}N)$ where $\mathcal{P}_{Q_0,Q_1}N := \{\gamma \in C^0([0,1], N) | \gamma(0) \in Q_0, \gamma(1) \in Q_1\}$ the path space from Q_0 to Q_1 on N.

For a concrete example which will be used in our applications, we need the following topological Lemma. This Lemma slightly generalize the Lemma in Kang's paper [30] and the proof is almost same.

Lemma 5.2.8. *Let N be a closed connected manifold and Q_0, Q_1 be connected submanifolds of N. If Q_0 and Q_1 are contractible in N, then we have the following homotopy equivalence*

$$\mathcal{P}_{Q_0,Q_1}N \simeq \Omega N \times Q_0 \times Q_1$$

where ΩN is the based loop space of N.

Proof. We pick a point $p \in N$ and fix the following homotopies

$$F_0 : [0,1] \times Q_0 \to N, \quad F_1 : [0,1] \times Q_1 \to N$$

such that

$$F_0(0,*) = p, \quad F_0(1,*) = i_{Q_0} \quad \text{and} \quad F_1(0,*) = p, \quad F_1(1,*) = i_{Q_1}$$

where p denote the constant map to the point p and i_{Q_i} is the canonical inclusion for the submanifold $Q_i \subset N$. We define maps $\mathcal{E}$ and $\mathcal{F}$ by

$$\begin{array}{rcl} \mathcal{E} : \mathcal{P}_{Q_0,Q_1}N & \to & \Omega N \times Q_0 \times Q_1, \\ v & \mapsto & (\overline{F_1^{v(1)}}\#v\#F_0^{v(0)}, v(0), v(1)) \end{array}$$

and

$$\begin{array}{rcl} \mathcal{F} : \Omega N \times Q_0 \times Q_1 & \to & \mathcal{P}_{Q_0,Q_1}N, \\ (w,x,y) & \mapsto & (F_1^y\#w\#\overline{F_0^x}) \end{array}$$

where $F_i^x(t) := F_i(t,x)$ is a curve obtained by fixing a point in Q_i and overline means the curve of opposite direction. Then we have that

$$\mathcal{F} \circ \mathcal{E}(v) = F_1^{v(1)}\#\overline{F_1^{v(1)}}\#v\#F_0^{v(0)}\#\overline{F_0^{v(0)}}$$

and

$$\mathcal{E} \circ \mathcal{F}(w,x,y) = (\overline{F_1^{v(1)}}\#F_1^{v(1)}\#w\#\overline{F_0^{v(0)}}\#F_0^{v(0)}, x, y)$$

in composition. It is easy to see that $\mathcal{F} \circ \mathcal{E}$ is homotopic to the identity on $\mathcal{P}_{Q_0,Q_1}N$ by contracting the trivial paths $F_1^{v(1)}\#\overline{F_1^{v(1)}}$, $F_0^{v(0)}\#\overline{F_0^{v(0)}}$ and reparametrizing time. Similarly, we can show that $\mathcal{E} \circ \mathcal{F}$ is homotopic to the identity on $\Omega N \times Q_0 \times Q_1$. This completes the proof of Lemma 5.2.8. $\square$

Example 5.2.9. We consider the cotangent space $(T^*S^2, \omega_{can} = d\lambda_{can})$ of S^2. Let $Q_1 = \{(0, x_2, x_3)\}$ and $Q_2 = \{(x_1, 0, x_3)\}$ be equators of $S^2 := \{x \in$

$\mathbb{R}^3||x| = 1\}$. We have shown in Example 5.2.3 that $\mathrm{Fix}(\mathcal{R}_i) = \nu^*Q_i$ for $i = 1, 2$. First, we consider a fiberwise star-shaped domain $M \subset T^*S^2$ such that M is invariant under $\mathcal{R}_1$. The triple $(M, \lambda_{can}, \mathcal{R}_1)$ defines a real Liouville domain. Thus we obtain a symmetric orbit whenever we have a Reeb chord of length T from $\nu^*Q_1 \cap \partial M$ to $\nu^*Q_1 \cap \partial M$. From Example 5.2.7, we can compute the wrapped Floer homology

$$WFH_*(\nu^*Q_1; M) \cong H_*(\mathcal{P}_{Q_1,Q_1}S^2)$$

of M with the Lagrangian ν^*Q_1. Using Lemma 5.2.8, we have that

$$\mathcal{P}_{Q_1,Q_1}S^2 \simeq \Omega S^2 \times S^1 \times S^1.$$

Therefore we obtain the explicit form

$$WFH_*(\nu^*Q_1; M) = \begin{cases} \mathbb{Z}_2 & \text{for} \quad * = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * \geq 2. \end{cases}$$

of the wrapped Floer homology. Here, we use the fact that $H_*(\Omega S^2; \mathbb{Z}_2) = \mathbb{Z}_2$ for all $* = 0, 1, 2, \cdots$. Second, we consider a fiberwise star-shaped domain $M \subset T^*S^2$ such that M is invariant under $\mathcal{R}_1$ and $\mathcal{R}_2$. If we have a length T Reeb chord, say w, from $\nu^*Q_1 \cap \partial M$ to $\nu^*Q_2 \cap \partial M$, then $w \# \mathcal{R}_{2,*}w$ is a length $2T$ Reeb chord from $\nu^*Q_1 \cap \partial M$ to $\nu^*Q_1 \cap \partial M$. Thus w corresponds to a doubly symmetric periodic Reeb orbit on ∂M. Using Example 5.2.7 and Lemma 5.2.8 again, we have that

$$WFH_*(\nu^*Q_1, \nu^*Q_2; M) \cong H_*(\mathcal{P}_{Q_1,Q_2}S^2).$$

and hence

$$WFH_*(\nu^*Q_1, \nu^*Q_2; M) = \begin{cases} \mathbb{Z}_2 & \text{for} \quad * = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * \geq 2. \end{cases}$$

As a result of the computations in Example 5.2.9, we get the following Corollary.

Corollary 5.2.10. *Suppose U is an open subset of $T^*\mathbb{R}^2$. Assume that $H : U \to \mathbb{R}$ is fiberwise star-shaped at energy c.*
(1) If H is invariant under $\mathcal{R}_1$, then there is a symmetric orbit on the regularized energy hypersurface at c.
(2) If H is invariant under $\mathcal{R}_1$ and $\mathcal{R}_2$, then there is a doubly symmetric orbit on the regularized energy hypersurface at c.

If we apply this Corollary to the restricted three body problem and Hill's lunar problem, then we can give a homological reason of the classical results of Birkhoff. Using the shooting method in [12], one can prove the existence of a symmetric orbit of the restricted three body problem and the existence of a doubly symmetric orbit of Hill's lunar problem. This proof excludes the collision orbit case and so in this sense Birkhoff's result is stronger than Corollary 5.2.10. Moreover, Birkhoff's method guarantees the simpleness of such orbits when we project to the Hill's region but this homological reason does not.

Chapter 6

Spectral invariants for fiberwise star-shaped domains in cotangent bundles

In this chapter, we will define symplectic capacity for fiberwise star-shaped domains in a cotangent bundle using spectral invariants of symplectic homology and wrapped Floer homology. We will define the spectral invariant using usual min-max argument. However we will assign the number to each fiberwise star-shaped domains in a fixed cotangent bundles. That will help to see the embeddings among them. In section 6.1, we will define spectral invariant in symplectic homologies of fiberwise star-shaped domains in a cotangent bundle. In section 6.2, we will define spectral invariant in wrapped Floer homologies of fiberwise star-shaped domains in a cotangent bundle.

6.1 Spectral invariant in symplectic homology

Let N be a closed manifold. Consider the cotangent bundle $(T^*N, d\lambda_{can})$ with the canonical symplectic structure. We define a symplectic capacity for fiberwise star-shaped domains in T^*N. Let M be a fiberwise star-shaped domain.

Then $(M, \omega = d\lambda_{can}|_M)$ is a Liouville domain as we discussed in Example 5.1.2. We note that $[\omega]|_{\pi_2(M)} = 0$, $c_1(M)|_{\pi_2(M)} = 0$ and the symplectic completion $\hat{M}$ can be regarded as T^*N. The symplectic homology for (M, ω_{can}) is isomorphic to the homology of $H_*(\Lambda N)$ by Example 5.1.11. We will denote this isomorphism by

$$\Psi_M : H_*(\Lambda N) \to SH_*(M).$$

Recall the long exact sequence of symplectic homology (5.1.7)

$$\cdots \to SH_*^{<b}(M) \xrightarrow{i_M^b} SH_*(M) \xrightarrow{j_M^b} SH_*^{\geq b}(M) \to SH_{*-1}^{<b}(M) \xrightarrow{i_M^b} \cdots$$

for each $b \in \mathbb{R}$. Using this long exact sequence, we assign a constant in the following way.

Definition 6.1.1. In the above setup, we define

$$c_N(M, \alpha) := \inf\{b \in \mathbb{R} \cup \{+\infty\} | \Psi_M(\alpha) \in \mathrm{im}(i_M^b)\}$$

for a homology class $0 \neq \alpha \in H_*(\Lambda N)$. This constant $c_N(M, \alpha)$ is called the *spectral invariant of α in the symplectic homology of M.*

Remark 6.1.2. One can see immediately that we have another description of the spectral invariant c_N. Let us define a constant

$$c'_N(M, \alpha) := \sup\{b \in \mathbb{R} \cup \{+\infty\} | j_M^b(\Psi_M(\alpha)) \neq 0\}$$

for a while. For any $\epsilon > 0$, there exist $b \in [c_N(M, \alpha), c_N(M, \alpha) + \epsilon)$ and $\sigma \in SH_*^{>b}(M)$ such that $\Psi_M(\alpha) = i_M^b(\sigma)$. Then we have $j_M^b(\Psi_M(\alpha)) = j_M^b \circ i_M^b(\sigma) = 0$ by exactness. This implies that $c'_N(M, \alpha) \leq b$ and so $c'_N(M, \alpha) \leq c_N(M, \alpha)$ because ϵ is arbitrary. On the other hand, for any $b > c'_N(M, \alpha)$, we have $j_M^b(\Psi_M(\alpha)) = 0$. Then we have $\Psi_M(\alpha) \in \ker j_M^b = \mathrm{im} i_M^b$. This implies that $c_N(M, \alpha) \leq b$ and so $c_N(M, \alpha) \leq c'_N(M, \alpha)$. This proves $c_N(M, \alpha) = c'_N(M, \alpha)$. Thus we will denote this common value by $c_N(M, \alpha)$.

Because we have a constant whenever we have a fiberwise star-shaped domain and a homology class of the free loop space ΛN, we can think of c_N as a map

$$c_N : FSD(N) \times H_*(\Lambda N)^\times \to \mathbb{R}$$

where $FSD(N)$ is the *set of all fiberwise star-shaped domains on* T^*N and $H_*(\Lambda N)^\times = H_*(\Lambda N)\backslash\{0\}$. We will prove the following properties of c_N.

Theorem A1 (Properties of c_N)**.** The map

$$c_N : FSD(N) \times H_*(\Lambda N)^\times \to \mathbb{R}, \quad (M, \alpha) \mapsto c(M, \alpha)$$

satisfies the following properties.

1) (Conformality) $c_N(kM, \alpha) = kc_N(M, \alpha)$ for all $k \in \mathbb{R}^+$.

2) (Monotonicity) $c_N(M_2, \alpha) \geq \kappa_{min}(\Sigma_1, \Sigma_2)c_N(M_1, \alpha)$ for all $M_1, M_2 \in FSD(N)$ where $\Sigma_i = \partial M_i$, $i = 1, 2$ and $\kappa_{min}(\Sigma_1, \Sigma_2) = \min_{x\in\Sigma_1}\{\kappa(x)|\kappa(x)x \in \Sigma_2, \kappa(x) > 0\}$.

3) (Spectrality) $c_N(M, \alpha) \in Spec(\Sigma, \lambda_{can})$ where $\Sigma = \partial M$.

for each $\alpha \in H_*(\Lambda N)^\times$.

In Theorem A1, kM in 1) denotes the Liouville domain obtained by multiplying k on each fiber of M as a scalar multiplication in each cotangent space. We define $\kappa_{min}(\Sigma_1, \Sigma_2)$ in 2) by

$$\kappa_{min}(\Sigma_1, \Sigma_2) = \min_{x\in\Sigma_1}\{\kappa(x)|\kappa(x)x \in \Sigma_2, \kappa(x) > 0\}$$

and we define similarly

$$\kappa_{max}(\Sigma_1, \Sigma_2) = \max_{x\in\Sigma_1}\{\kappa(x)|\kappa(x)x \in \Sigma_2, \kappa(x) > 0\}$$

for any pair of fiberwise star-shaped hypersurfaces Σ_1, Σ_2. Clearly, these numbers are positive. Finally, We denote by $\mathcal{P}(\Sigma, \lambda_{can})$ the set of all periodic Reeb orbits of the contact manifold (Σ, λ_{can}). As we discussed in section 5.1, we can think of the Reeb orbit as a generator of symplectic homology. We denote by $Spec(\Sigma, \lambda_{can}) \subset \mathbb{R}$ the set of all nonnegative Reeb periods of the contact manifold (Σ, λ_{can}). The period of a periodic Reeb orbit can be regarded as an action value of the Reeb orbit in symplectic homology.

We will prove Theorem A1 in this section. For the proof, we need the following Lemmas.

Lemma 6.1.3. *Let M be a fiberwise star-shaped domain in T^*N. If $b \in \mathbb{R}^+ \backslash Spec(\Sigma, \lambda_{can})$, then we have an isomorphism*

$$SH_*^{<b}(M) \simeq FH_*(K_M^b)$$

between the symplectic homology of action less than b and the Floer homology with Hamiltonian $K_M^b : \hat{M} = T^*N \to \mathbb{R}$. The Hamiltonian K_M^b is given by

$$K_M^b(x) = \begin{cases} 0 & \text{if} \quad x \in M, \\ b(r-1) & \text{if} \quad x = (r,p) \in [0, +\infty) \times \Sigma \end{cases}$$

Proof. By definition of the symplectic homology of M, we have

$$SH_*^{<b}(M) = \varinjlim_{H \in Ad(M)} FH_*^{<b}(H).$$

Since the action functional $\mathcal{A}_{K_M^b}$ has no critical value larger than b, we have

$$FH_*(K_M^b) \simeq FH_*^{<b}(K_M^b) \simeq FH_*^{<b}(K_M^c)$$

for all $c \geq b$. Since the set of functions $\{K_M^c | c \geq b\}$ is cofinal in $Ad(M)$, we have

$$SH_*^{<b}(M) = \varinjlim_{c \geq b} FH_*^{<b}(K_M^c) = FH_*(K_M^b).$$

This proves Lemma 6.1.3. □

Throughout this section, we will assume that $b \in \mathbb{R}^+ \backslash Spec(\Sigma, \lambda_{can})$. Because it is known that $Spec(\Sigma, \lambda_{can})$ is discrete for a generic Σ.

Lemma 6.1.4. *The following diagram*

$$\begin{tikzcd} & FH_*(K_M^b) \arrow[dl, "\phi^{<b}_{K_M^b}"', "\simeq"] \arrow[d, "\phi_{K_M^b}"] \\ SH_*^{<b}(M) \arrow[r, "i_M^b"'] & SH_*(M) \end{tikzcd}$$

commutes where ϕ is the canonical inclusion in the direct system from a Floer homology of M to the symplectic homology of M.

Proof. For an admissible Hamiltonian $H \in Ad(M)$, we have the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & FH_*^{<b}(H) \arrow[r] \arrow[d, "\phi_H^{<b}"'] & FH_*(H) \arrow[r] \arrow[d, "\phi_H"'] & FH_*^{\geq b}(H) \arrow[r] \arrow[d, "\phi_H^{\geq b}"'] & FH_{*-1}^{<b}(H) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & SH_*^{<b}(M) \arrow[r, "i_M^b"'] & SH_*(M) \arrow[r, "j_M^b"'] & SH_*^{\geq b}(M) \arrow[r, "\partial"'] & SH_{*-1}^{<b}(M) \arrow[r] & \cdots \end{tikzcd}$$

We focus on the first square of the above commutative diagram and we replace H by K_M^b. Then we have the commutative diagram

$$\begin{tikzcd} FH_*^{<b}(K_M^b) \arrow[r, "\simeq"'] \arrow[d, "\phi^{<b}_{K_M^b}"'] & FH_*(K_M^b) \arrow[d, "\phi_{K_M^b}"'] \\ SH_*^{<b}(M) \arrow[r, "i_M^b"'] & SH_*(M) \end{tikzcd}$$

with an isomorphism on the upper and right sides by Lemma 6.1.3. If we identify two Floer homology groups in the first row, then we get the desired commutative diagram. This proves Lemma 6.1.4. □

Remark 6.1.5. By virtue of Lemma 6.1.3 and 6.1.4, we can identify the induced map $SH_*^{<b}(M) \xrightarrow{i_M^b} SH^*(M)$ on the symplectic homology with the canonical map $FH_*(K_M^b) \xrightarrow{\phi_{K_M^b}} SH_*(M)$ of the direct system.

First, we prove Theorem A1 3). This will be done by proving the following Lemma.

Lemma 6.1.6. *For each $M \in FSD(N)$ and $\alpha \in H_*(\Lambda N)$,*

$$c_N(M,\alpha) = \min_{\sum_{x\in\mathcal{P}(\Sigma,\lambda)} c_x x \in \Psi_M(\alpha)} \max\left\{\int_{S^1} x^*\lambda_{can} | c_x \neq 0\right\}$$

Proof. Let us denote that

$$\bar{c}(M,\alpha) = \min_{\sum_{x\in\mathcal{P}(\Sigma,\lambda)} c_x x \in \Psi_M(\alpha)} \max\left\{\int_{S^1} x^*\lambda_{can} | c_x \neq 0\right\}$$

for a moment. We want to show that $c_N(\Sigma,\alpha) = \bar{c}(\Sigma,\alpha)$. Let $\sigma = \sum_{x\in\mathcal{P}(\Sigma,\lambda)} c_x x \in \Psi_M(\alpha)$ be a representative of symplectic homology where the infimum is achieved, that is, $\max\left\{\int_{S^1} x^*\lambda_{can} | c_x \neq 0\right\} = \bar{c}(\Sigma,\alpha) =: \bar{c}$. For any $\epsilon > 0$, if we take $b = \bar{c} + \epsilon$, then $[\sigma] \in FH_*(K^b_M)$ since every generator of action below b in chain complex and $\partial\sigma = 0$ as well. By the choice of σ, we have $i^b_M(\sigma) = \Psi_M(\alpha)$ and this implies $c_N(M,\alpha) \leq b$ and so $c_N(M,\alpha) \leq \bar{c}$ because ϵ is arbitrary.

Conversely, we suppose that $b < \bar{c}$ and $\Psi_M(\alpha) \in \mathrm{im}(i^b_M)$. Then there exists $\sigma \in FH_*(K^b_M)$ such that $i^b_M(\sigma) = \Psi_M(\alpha)$. Since $i^b_M(\sigma)$ consists of the Reeb orbits whose periods are less than or equal to b. This implies $\bar{c} \leq b$. This contradicts the assumption. Therefore, the inequality $c_N(M,\alpha) \geq \bar{c}$. This completes the proof of Lemma 6.1.6. □

Remark 6.1.7. One can regard $c_N(M,\alpha)$ as the spectral invariant corresponding to α for the Floer homology of Hamiltonian $K^b_M : T^*N \to \mathbb{R}$ of sufficiently large b. In that reason, we call $c_N(M,\alpha)$ by the spectral invariant of α in the symplectic homology of M.

We consider two fiberwise star-shaped domains $M_1, M_2 \in FSD(N)$ in T^*N. If we assume that $\kappa_{min}(\Sigma_1,\Sigma_2) \geq 1$, that is $M_1 \subset M_2$ and abbreviate $\kappa_{min} := \kappa_{min}(\Sigma_1,\Sigma_2)$, then one can easily see that

$$K^{b\kappa_{min}}_{M_2}(x) \leq K^b_{M_1}(x)$$

for all $x \in T^*N = \hat{M}_1 = \hat{M}_2$. Thus we have the monotone homomorphism

$$\phi_{(K^b_{M_1}, K^{b\kappa_{min}}_{M_2})} : FH_*(K^{b\kappa_{min}}_{M_2}) \to FH_*(K^b_{M_1})$$

between Floer homologies. Using this morphism, we have the following Lemma.

Lemma 6.1.8. *Let M_1, M_2 be fiberwise star-shaped domain in T^*N. Suppose that $M_1 \subset M_2$. Then the following diagram commutes.*

$$\begin{tikzcd}
FH_*(K^{b\kappa_{min}}_{M_2}) \arrow[r, "\phi_{K^{b\kappa_{min}}_{M_2}}"'] \arrow[dd, "\phi_{(K^b_{M_1}, K^{b\kappa_{min}}_{M_2})}"'] & SH_*(M_2) & \\
 & & H_*(\Lambda N) \arrow[ul, "\Psi_{M_2}"'] \arrow[dl, "\Psi_{M_1}"] \\
FH_*(K^b_{M_1}) \arrow[r, "\phi_{K^b_{M_1}}"'] & SH_*(M_1) &
\end{tikzcd}$$

Proof. We recall the isomorphism between $\Psi_M : H_*(\Lambda N) \to SH_*(M)$ in [1]. They constructed the isomorphism

$$\Theta^{AS}_H : FH_*(H) \to HM_*(\mathcal{E}_L)$$

between a Floer homology of a quadratic Hamiltonian $H \in Qd(T^*N)$ and a Morse homology of Lagrangian action functional for $L = \mathcal{L}(H)$, Legendre transformation of H. After this construction, Ψ_M can be obtained by identifying $SH_*(M)$ with $FH_*(H)$ and $HM_*(\mathcal{E}_L)$ with $H_*(\Lambda N)$. We can take a quadratic Hamiltonian $H \in Qd(T^*N)$ on T^*N such that $H \geq K^{b\lambda_{min}}_{M_2}$ and $H \geq K^b_{M_1}$. For example, we fix a metric g on N and take sufficiently large s such that $H(q,p) = s|p|^2_g$ satisfies the above inequalities. Then we have

$$\phi_{(H, K^b_{M_1})} \circ \phi_{(K^b_{M_1}, K^{b\kappa_{min}}_{M_2})} = \phi_{(H, K^{b\kappa_{min}}_{M_2})}$$

by the naturality of monotone homomorphism. In order to use the usual compactness argument in symplectic homology, we can replace H by the function that coincides with H in $\{|p|_g < R\}$ for sufficiently large R and is asymptotically linear at infinity. Thus we have

$$\Theta_H^{AS} \circ \phi_{(H,K^b_{M_1})} \circ \phi_{(K^b_{M_1},K^{b\kappa_{min}}_{M_2})} = \Theta_H^{AS} \circ \phi_{(H,K^{b\kappa_{min}}_{M_2})}$$

This implies that

$$\Psi_{M_1}^{-1} \circ \phi_{K^b_{M_1}} \circ \phi_{(K^b_{M_1},K^{b\kappa_{min}}_{M_2})} = \Psi_{M_2}^{-1} \circ \phi_{K^{b\kappa_{min}}_{M_2}}$$

from the following commutative diagram.

$$\begin{tikzcd} SH_*(M) \arrow[d, "\cong"'] & H_*(\Lambda N) \arrow[l, "\Psi_M"'] \arrow[d, "\cong"'] \\ FH_*(H) \arrow[r, "\Theta_H^{AS}"'] & HM_*(\mathcal{E}_L) \end{tikzcd}$$

This proves Lemma 6.1.8. □

Lemma 6.1.8 implies the following crucial fact.

$$\begin{aligned} & \Psi_{M_2}(\alpha) \in \mathrm{im}(\phi_{K^{b\kappa_{min}}_{M_2}}) \\ \Longleftrightarrow \quad & \Psi_{M_2}(\alpha) \in \mathrm{im}(\Psi_{M_2} \circ \Psi_{M_1}^{-1} \circ \phi_{K^b_{M_1}} \circ \phi_{(K^b_{M_1},K^{b\kappa_{min}}_{M_2})}) \\ \Longleftrightarrow \quad & \alpha \in \mathrm{im}(\Psi_{M_1}^{-1} \circ \phi_{K^b_{M_1}} \circ \phi_{(K^b_{M_1},K^{b\kappa_{min}}_{M_2})}) \\ \Longleftrightarrow \quad & \Psi_{M_1}(\alpha) \in \mathrm{im}(\phi_{K^b_{M_1}} \circ \phi_{(K^b_{M_1},K^{b\kappa_{min}}_{M_2})}) \\ \Longrightarrow \quad & \Psi_{M_1}(\alpha) \in \mathrm{im}(\phi_{K^b_{M_1}}) \end{aligned}$$

for any $\alpha \in H_*(\Lambda N)^\times$ and $b \in \mathbb{R}$. In sum, we have

$$\Psi_{M_2}(\alpha) \in \mathrm{im}(\phi_{K^{b\kappa_{min}}_{M_2}}) \Longrightarrow \Psi_{M_1}(\alpha) \in \mathrm{im}(\phi_{K^b_{M_1}})$$

for any $\alpha \in H_*(\Lambda N)^\times$, $b \in \mathbb{R}$ and $M_1, M_2 \in FSD(N)$. Therefore, we have proven the following Lemma.

Lemma 6.1.9. *Let M_1, M_2 be fiberwise star-shaped domains in T^*N. Sup-*

pose that $\kappa_{min}(\Sigma_1, \Sigma_2) \geq 1$ *for* $\Sigma_i = \partial M_i (i = 1, 2)$*. Then the inequality*

$$c_N(M_2, \alpha) \geq \kappa_{min}(\Sigma_1, \Sigma_2) c_N(M_1, \alpha)$$

holds for each $\alpha \in H_*(\Lambda N)$*. In particular, the above inequality implies*

$$c_N(M_2, \alpha) \geq c_N(M_1, \alpha)$$

provided $\kappa_{min}(\Sigma_1, \Sigma_2) \geq 1$*, that is,* $M_1 \subset M_2$*.*

We want to extend the above Lemma for any $\kappa_{min}(\Sigma_1, \Sigma_2) \in \mathbb{R}^+$. We need the following Lemma in order to define a contactomorphism between two fiberwise star-shaped hypersurfaces.

Lemma 6.1.10. *Let* M_1, M_2 *be fiberwise star-shaped domains in* T^*N*. For* $\Sigma_1 = \partial M_1$ *and* $\Sigma_2 = \partial M_2$*, we define a function*

$$f_{\Sigma_1}^{\Sigma_2} : \Sigma_1 \to \mathbb{R}^+, \quad f_{\Sigma_1}^{\Sigma_2}(x) \cdot x \in \Sigma_2$$

on Σ_1*. In local coordinates* $x = (q, p)$, $f_{\Sigma_1}^{\Sigma_2}(x) \cdot x = f_{\Sigma_1}^{\Sigma_2}(x) \cdot (q, p) = (q, f_{\Sigma_1}^{\Sigma_2}(x) p)$ *is the scalar multiplication on the cotangent space. We define a diffeomorphism*

$$F_{\Sigma_1}^{\Sigma_2} : \Sigma_1 \to \Sigma_2, \quad x \mapsto f_{\Sigma_1}^{\Sigma_2}(x) \cdot x$$

from Σ_1 *to* Σ_2*. Then the map* $F_{\Sigma_1}^{\Sigma_2}$ *is a contactomorphism between* (Σ_1, ξ_{can}) *and* (Σ_2, ξ_{can}) *where* $\xi_{can} = \ker \lambda_{can}$*. More precisely, the pull-back of the Liouville 1-form* λ_{can} *is given by*

$$(F_{\Sigma_1}^{\Sigma_2})^* \lambda_{can} = f_{\Sigma_1}^{\Sigma_2} \cdot \lambda_{can}.$$

Proof. It suffices to prove the last statement. We recall the canonical 1-form $\lambda_{can} = pdq$ in the local coordinate $x = (q, p)$. We will directly compute the evaluation of the pull-back form $(F_{\Sigma_1}^{\Sigma_2})^* \lambda_{can}(x)$ for an arbitrary tangent vector $h \in T_x T^*N$ for $x \in \Sigma_1$. Assume that $h = h_q + h_p$ where $h_q \in < \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \cdots, \frac{\partial}{\partial q_n} >$ and $h_p \in < \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \cdots, \frac{\partial}{\partial p_n} >$. For a notational convenience, we denote $F := F_{\Sigma_1}^{\Sigma_2}$ and $f := f_{\Sigma_1}^{\Sigma_2}$ in this proof.

$$\begin{aligned}
F^*\lambda_{can}(x)(h) &= \lambda_{can}(F(x))(DF(h)) \\
&= \lambda_{can}((q, f(x)p))(h_q + f(x)h_p + df(x)(h) \cdot p\frac{\partial}{\partial p}) \\
&= \lambda_{can}((q, f(x)p))(h_q) \\
&= \lambda_{can}((q, f(x)p))(h) \\
&= f(x)\lambda_{can}((q, p))(h)
\end{aligned}$$

Therefore, we have $(F_{\Sigma_1}^{\Sigma_2})^*\lambda_{can} = f_{\Sigma_1}^{\Sigma_2} \cdot \lambda_{can}$ and this proves Lemma 6.1.10. □

Remark 6.1.11. Lemma 6.1.10 implies that the map

$$F_{\Sigma_1}^{\Sigma_2} : (\Sigma_1, f_{\Sigma_1}^{\Sigma_2} \cdot \lambda_{can}) \to (\Sigma_2, \lambda_{can})$$

is a strict contactomorphism for all pair of fiberwise star-shaped hypersurfaces Σ_1, Σ_2. In particular, if $\Sigma_2 = k\Sigma_1$ for some $k > 0$, then we have a strict contactomorphism

$$F : (\Sigma_1, k\lambda_{can}) \to (\Sigma_2, \lambda_{can})$$

and this extends to a symplectomorphism between two Liouville domains $(M_1, k\omega_{can}), (M_2, \omega_{can})$ enclosed by Σ_1, Σ_2, respectively. This implies the conformality of c_N as follows.

$$c_N(kM_1, \alpha) = c_N(M_2, \alpha) = kc_N(M_1, \alpha)$$

This proves Theorem A1 1).

We can prove Theorem A1 2) by combining Lemma 6.1.9 and Theorem A1 1). Let M_1, M_2 be fiberwise star-shaped domains in T^*N. We denote $k = \kappa_{min}(\Sigma_1, \Sigma_2)$ where $\Sigma_i = \partial M_i$ (i=1,2). If $k \geq 1$, then we already have that $c_N(M_2, \alpha) \geq \kappa_{min}(\Sigma_1, \Sigma_2)c_N(M_1, \alpha)$ from Lemma 6.1.9. Suppose that $0 < k < 1$. If we consider kM_1 instead of M_1, then $\kappa_{min}(k\Sigma_1, \Sigma_2) = 1$. Hence we can apply Lemma 6.1.9 for the pair (kM_1, M_2) and so we have

$$c_N(M_2, \alpha) \geq c_N(kM_1, \alpha) = kc_N(M_1, \alpha) = \kappa_{min}(\Sigma_1, \Sigma_2)c_N(M_1, \alpha)$$

using Theorem A1 1). This proves Theorem A1 2). This completes the proof of Theorem A1.

We have proven Theorem A1. Therefore, as we mentioned, the spectral invariant $c_N(\cdot, \alpha)$ of α can play the role of symplectic capacity for $FSD(N)$ provided $c_N(\cdot, \alpha) \neq 0$. Moreover, by spectrality of Theorem A1, the spectral invariant $c_N(M, \alpha)$ of α in the symplectic homology of M should be one of Reeb periods. We will use the spectral invariant c_{S^2} to obtain estimates action values of Hill's lunar problem in chapter 7.

6.2 Spectral invariant in wrapped Floer homology

Let N be a closed manifold. Consider the cotangent manifold $(T^*N, d\lambda_{can})$ with the canonical symplectic form. Let Q_0 and Q_1 be submanifolds of N. Their conormal bundles ν^*Q_0 and ν^*Q_1 are Lagrangian submanifolds of T^*N, respectively. If we have a fiberwise star-shaped domain M of T^*N, then $(M, \omega_{can} = d\lambda_{can})$ becomes a Liouville domain. Moreover, the intersections $\mathcal{L}_i = M \cap N^*Q_i$ define Lagrangian submanifolds in M. We will assume that this pair of Lagrangians $(\mathcal{L}_0, \mathcal{L}_1)$ is an admissible pair in M. We will define a spectral invariant of wrapped Floer homology for $(\mathcal{L}_0, \mathcal{L}_1; M\omega_{can})$. We have seen that the wrapped Floer homology for $(\mathcal{L}_0, \mathcal{L}_1; M, \omega_{can})$ is isomorphic to the homology of $H_*(\mathcal{P}_{Q_0,Q_1}N)$. We will denote this isomorphism by

$$\Psi_M : H_*(\mathcal{P}_{Q_0,Q_1}N) \to WFH_*(\mathcal{L}_0, \mathcal{L}_1, M)$$

for each fiberwise star-shaped domain $M \subset T^*N$. We recall the long exact sequence of wrapped Floer homology (5.2.4). For $b \in \mathbb{R} \cup \{+\infty\}$, we have

$$\begin{aligned}\cdots \;\to\; & WFH_*^{<b}(\mathcal{L}_0, \mathcal{L}_1, M) \xrightarrow{i_M^b} WFH_*(\mathcal{L}_0, \mathcal{L}_1, M) \xrightarrow{j_M^b} WFH_*^{\geq b}(\mathcal{L}_0, \mathcal{L}_1, M) \\ \to\; & WFH_{*-1}^{<b}(\mathcal{L}_0, \mathcal{L}_1, M) \xrightarrow{i_M^b} \cdots\end{aligned}$$

Definition 6.2.1. In the above setup, we define

$$c_{Q_0,Q_1,N}(M, \alpha) := \inf\{b \in \mathbb{R} \cup \{+\infty\} | \Psi_M(\alpha) \in \mathrm{im}(i_M^b)\}$$

for a homology class $0 \neq \alpha \in H_*(\mathcal{P}_{Q_0,Q_1}N)$. This constant $c_{Q_0,Q_1,N}(M, \alpha)$ is called the *spectral invariant of* α *in the wrapped Floer homology of* $(\mathcal{L}_0, \mathcal{L}_1, M)$. We denote by $c_{q,N}$ when $Q_0 = Q_1 = Q$.

Following the proof in section 6.1, we can prove the following Theorem.

Theorem A2 (Properties of $c_{Q_0,Q_1,N}$)**.** The map

$$c_{Q_0,Q_1,N} : FSD(N) \times H_*(\mathcal{P}_{Q_0,Q_1}N)^\times \to \mathbb{R}, \quad (M, \alpha) \mapsto c_{Q_0,Q_1,N}(M, \alpha)$$

satisfies the following properties.

1) (Conformality) $c_{Q_0,Q_1,N}(kM, \alpha) = kc_{Q_0,Q_1,N}(M, \alpha)$ for all $k \in \mathbb{R}^+$.

2) (Monotonicity) $c_{Q_0,Q_1,N}(M_2, \alpha) \geq \kappa_{min}(\Sigma_1, \Sigma_2)c_{Q_0,Q_1,N}(M_1, \alpha)$ for all $M_1, M_2 \in FSD(N)$ where $\Sigma_i = \partial M_i$, $i = 1, 2$ and $\kappa_{min}(\Sigma_1, \Sigma_2) = \min_{x\in\Sigma_1}\{\kappa(x)|\kappa(x)x \in \Sigma_2, \kappa(x) > 0\}$.

3) (Spectrality) $c_{Q_0,Q_1,N}(M, \alpha) \in Spec(\Sigma, \lambda_{can}; \partial\mathcal{L}_0, \partial\mathcal{L}_1)$ where $\Sigma = \partial M$ and $\mathcal{L}_i = M \cap N^*Q_i$.

for each $\alpha \in H_*(\mathcal{P}_{Q_0,Q_1}N)^\times$.

The proof is an analogue of the proof of Theorem A1. Hence, instead of the proof of Theorem A2, examples of the spectral invariant in wrapped Floer homology for applications will be given in this section.

Example 6.2.2 (real Liouville domain structures on cotangent bundles)**.** Let N be a closed manifold. Consider an involution $i : N \to N$. The canonically induced map $\mathcal{I} : T^*N \to T^*N$ on the cotangent bundle space is a symplectomorphism of $(T^*N, d\lambda_{can})$. Consider the map $r : T^*N \to T^*N$, $r(q,p) = (q,-p)$. Note that this map is an anti-symplectic involution and $Fix(r)$ is the zero section. If we define the composition map $\mathcal{R} := r \circ \mathcal{I} : T^*N \to T^*N$, the map $\mathcal{R}$ becomes an anti-symplectic involution of $(T^*N, d\lambda_{can})$. Assume that, in addition, $\mathcal{R}_*\lambda_{can} = -\lambda_{can}$. Let $M \subset T^*N$ be a fiberwise star-shaped domain of T^*N which is invariant under $\mathcal{R}$. Then the triple $(M, \lambda_{can}, \mathcal{R})$ defines a real Liouville domain. By Lemma 6.2.3 below, if $Fix(i) = Q$, then $Fix(\mathcal{R}) = \nu^*Q$. Moreover, by Theorem 2.4.11, there is a one-to-one correspondence between the set of all Reeb chords from $Fix(\mathcal{R}) \cap \partial M$ to itself and the set of all symmetric periodic orbits. Thus, the spectral invariant $c_{Q,N}(M,\alpha)$ in the wrapped Floer homology $WFH_*(\nu^*Q \cap M, M)$ provides information about $\mathcal{R}$-symmetric periodic orbits on $(\partial M, \lambda_{can}, \mathcal{R})$. This can play a role of *symmetric symplectic capacity* by choosing the suitable involution i under consideration.

Lemma 6.2.3. *Let $i, \mathcal{I}, r, \mathcal{R}$ be as above. Suppose that $Fix(i) = Q$ is a submanifold of N. Then $Fix(\mathcal{R}) = \nu^*Q$.*

Proof. The fixed point set

$$Fix(\mathcal{R}) = \{x \in T^*N | -\mathcal{I}(x) = x\}$$

of $\mathcal{R}$ has the form $\{(q,p) \in T^*N | (i(q), -i^*(p)) = (q,p)\}$ in the canonical local coordinate system induced from a local coordinate system on N. Therefore $(q,p) \in Fix(\mathcal{R})$ implies $q \in Q$. Fix $q \in Q$. Since $i^2 = id$, the differential $di(q) : T_q^*N \to T_q^*N$ has eigenvalue ± 1. In particular, we have that $di(q)|_{T_qQ} = id$ on T_qQ and $di(q)|_{N_qQ} = -id$ on the normal direction. This implies that

$$(q,p) \in Fix(\mathcal{R}) \iff p = -p \circ di(q) \iff p|_{T_qQ} = 0$$

for each $q \in Q$. It follows that $Fix(\mathcal{R}) = \nu^*Q$. This completes the proof of Lemma 6.2.3. $\square$

Chapter 7

Application to Hill's lunar problem

We defined spectral invariants in symplectic homology and wrapped Floer homology of fiberwise star-shaped domains in cotangent bundles. The spectral invariant can be used as (symmetric) symplectic capacities for fiberwise star-shaped domains in a cotangent bundle. In this chapter we will apply this to Hill's lunar problem. We need to compute spectral invariants of the periodic orbit and Lagrangian chords of the rotating Kepler problem. For this computation, it is important to know the Maslov indices and action values of periodic orbits and Lagrangian chords. We already discussed orbits in the rotating Kepler problem is given by composition of rotating flow and Hamiltonian flow of Kepler problem in section 3.2. In particular, the Conley-Zehnder indices of all periodic orbits of the rotating Kepler problem for energies below the critical value were completely determined in [7]. We will introduce the result briefly and we will compute the Maslov indices of Lagrangian chords in section 7.1. We will also compute the action value of each periodic orbit in section 7.2. This will lead us to derive the chain complex structure in the symplectic homology of the Liouville domains determined by the rotating Kepler problem. Using this information, we will compute in section 7.3 spectral invariants of the symplectic homology and the wrapped Floer homology of the Liouville domain determined by the rotating Kepler problem. We will derive some inclusion relations between the rotating Kepler problem and Hill's lunar problem in section 7.4. As results, we will get some estimates for spectral invariants of Hill's lunar problem in section 7.5 and these will be interpreted as estimates for the period of periodic orbits and symmetric periodic orbits.

7.1 The Maslov indices of the rotating Kepler problem

In this section, we will recall the result in [7]. They use the Conley-Zehnder index defined in [28]. Because the rotating Kepler problem is time-independent, there is the always present degeneracy if we use the definition in section 5.1. In [28], they use the restriction to the contact plane of hypersurface and so according to this definition, the retrograde and direct orbits are generically nondegenerate. In [7], they compute directly the indices of the retrograde and direct orbits using the suitably chosen trivialization of the contact structure. For the noncircular orbits $T_{k,l}$, the Conley-Zehnder indices are computed by using the fiberwise convexity of the regularized rotating Kepler problem. Because one can interpret the periodic orbits as critical points of the energy functional associated to a Finsler metric, the Conley-Zehnder index agrees with the Morse index of the energy functional. Then one can use the local invariance of Morse homology to determine the Morse index of $T_{k,l}$ at the bifurcation point. For example, if one has a degenerate orbit γ of S^1-family with the Conley-Zehnder index k, then this will become the nondegenerate obits γ^-, γ^+ of Conley-Zehnder index k and $k+1$, respectively, after a suitable perturbation, see [14] for the discussion of local Floer homology including the description of the perturbation. We will prove the following Theorem.

Theorem 7.1.1. *For any $c > \frac{3}{2}$, under the nondegenerate conditions, we have the following.*
(1) Let $\gamma^c_{R,N}$ and $\gamma^c_{D,N}$ be the N-th iteration of the retrograde and the direct

orbits, respectively, on Σ_R^c. Then

$$\mu_{CZ}(\gamma_{R,N}^c) = 1 + 2\left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1} \right\rfloor, \quad \mu_{CZ}(\gamma_{D,N}^c) = 1 + 2\left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \right\rfloor.$$

Moreover, $\mu_{CZ}(\gamma_{k,l}) = 2k-1$ for each $k > l \geq 1$ where $\gamma_{k,l}$ is a k-fold covered ellipse in an l-fold covered coordinate system.
(2) Let $v_{R,N}^c$ and $v_{D,N}^c$ be the N-th concatenation of the retrograde and the direct Lagrangian chords, respectively, from Fix$(\mathcal{R}_1)$ to itself. Then

$$\mu_{RS}(v_{R,N}^c) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1} \right\rfloor, \quad \mu_{RS}(v_{D,N}^c) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \right\rfloor$$

Moreover, $\mu_{RS}(v_{k,l}) = k$ for each $k > l \geq 1$ where $v_{k,l}$ is the chord corresponding to $\gamma_{k,l}$.
(3) Let $w_{R,N}^c$ and $w_{D,N}^c$ be the N-th concatenation of the retrograde and the direct Lagrangian chords, respectively, from Fix$(\mathcal{R}_1)$ to Fix$(\mathcal{R}_2)$. Then

$$\mu_{RS}(w_{R,N}^c) = 1 + \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}+1)} \right\rfloor, \quad \mu_{RS}(w_{D,N}^c) = 1 + \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}-1)} \right\rfloor$$

Moreover, $\mu_{RS}(w_{k,l}) = \frac{k}{2}$ for each $k > l \geq 1$ with $2|k$ and $2 \nmid l$ where $w_{k,l}$ is the chord corresponding to $\gamma_{k,l}$.
where $E = \frac{1}{2}|p|^2 - \frac{1}{|q|}$ for (q,p) is point on the trajectory.

We introduce the notations. We recall the planar Kepler problem introduced in Example 2.2.4. We denote the Hamiltonian H_{KP} by E for the notational convenience. As we have seen in Moser regularization in section 4.1, every orbit of the Hamiltonian equation for H_{KP} is periodic orbit, including the collision orbit after regularization, for negative energy. In fact, we know the orbits are either ellipses of eccentricity $\epsilon := \sqrt{2EL^2+1}$ or collision orbits by Kepler's laws of planetary motion. Moreover, we have the equality

$$T^2 = -\frac{\pi^2}{2E^3}$$

for the period T of the ellipse. The rotating Kepler problem is the Kepler problem in a rotating coordinate system. The Hamiltonian of the rotating Kepler problem is given by $H_{RKP} = E + L$ in our convention. We computed in section 3.2 that H_{RKP} has the unique critical value $-c_R^0 := -\frac{3}{2}$. We are interested in the energy hypersurfaces below this critical value. E, L are integrals of the rotating Kepler problem and so invariant under the Hamiltonian flow. Even though every orbit is periodic in the Kepler problem, not every orbit is periodic in the rotating Kepler problem. In fact, the Hamiltonian flow of the rotating Kepler problem is given by the composition

$$\phi_L^t \circ \phi_E^t$$

of two Hamiltonian flows where ϕ_L^t and ϕ_E^t are the flows generated by the Hamiltonians L and E.

First, the circular orbits in the Kepler problem give the circular orbits in the rotating Kepler problem and always give the periodic orbits in the rotating Kepler problem. By the direction of the rotation of circular orbits in the Kepler problem, we have two types of the circular orbits in the rotating Kepler problem. If we consider the opposite direction of the coordinate rotation, then we have the *retrograde orbit and denote by* γ_R. If we consider the same direction of rotation for orbit with the coordinate rotation, then we have the *direct orbit and denote by* γ_D. The circular orbits have the eccentricity $0 = \sqrt{2EL^2 + 1}$. If we fix an energy hypersurface $H_{RKP}^{-1}(-c)$, then we have the equation

$$0 = 2E(-c - E)^2 + 1$$

of the value E for the circular orbits. There exist two zeros less than $-\frac{1}{2}$ for each $c > \frac{3}{2}$. The smaller zero corresponds to the retrograde orbit and the other zero corresponds to the direct orbit.

Second, an ellipse orbit with positive eccentricity in the Kepler problem gives a periodic orbit in the rotating Kepler problem if and only if the period is a rational multiple of 2π. If the period of such a periodic orbit is $2\pi l$

for some $l \in \mathbb{N}$ and the orbit is a k-fold cover of ellipses in the inertial coordinate, then we we call this periodic orbit a *k-fold covered ellipse in an l-fold covered coordinate system and denote it by $\gamma_{k,l}$*. In the circular orbit case, there exist a retrograde orbit and a direct orbit for each $c > \frac{3}{2}$, up to time reparametrization. On the other hand, $\gamma_{k,l}$ exists only if energy condition is satisfied. We discuss the energy values where $\gamma_{k,l}$ exists for each k, l. From the definition of $\gamma_{k,l}$, the period of underlying ellipse in the Kepler problem is $T = \frac{2\pi l}{k}$. Using Kepler's law $T^2 = -\frac{\pi^2}{2E^3}$, we have

$$\frac{4\pi^2 l^2}{k^2} = -\frac{\pi^2}{2E^3}$$

and the energy level $E_{k,l}$ of this underlying ellipse of $\gamma_{k,l}$ is $E_{k,l} = -\frac{1}{2}(\frac{k}{l})^{\frac{2}{3}}$ for each $k, l \in \mathbb{N}$. In fact, we only consider the energy $E < \frac{-1}{2}$ and so we will assume $k > l$. From the eccentricity equation, $\gamma_{k,l}$ can exist only when the inequality

$$0 < 2E_{k,l}(c + E_{k,l})^2 + 1$$

holds. We solve this inequality for c. Then we have the energy range

$$c^-_{k,l} < c < c^+_{k,l}, \quad c^-_{k,l} := -E_{k,l} - \sqrt{\frac{1}{-2E_{k,l}}}, \quad c^+_{k,l} := -E_{k,l} + \sqrt{\frac{1}{-2E_{k,l}}}$$

for $\gamma_{k,l}$. At $c = c^-_{k,l}$, the eccentricity is 0 and $L = -E_{k,l} - c^+_{k,l} = \sqrt{\frac{1}{-2E_{k,l}}} > 0$. This means that $\gamma_{k,l}$ is the multiple cover of the retrograde orbit. In fact, the periodic orbit $\gamma_{k,l}$ degenerates to $(k + l)$-fold cover of the retrograde orbit at $c = c^-_{k,l}$. Similarly, the periodic orbit $\gamma_{k,l}$ degenerates to $(k - l)$-fold cover of the direct orbit at $c = c^+_{k,l}$. Using direct computation with suitably chosen trivialization, Morse index theory with fiberwise convexity of the rotating Kepler problem and bifurcation argument, they determined all Conley-Zehnder indices of above orbits. We recall the following result from [7]. This implies Theorem 7.1.1 (1).

Proposition 7.1.2 (Albers-Fish-Frauenfelder-van Koert)**.** *We define the N-*

th iteration of γ_R and γ_D by $\gamma_{R,N}$ and $\gamma_{D,N}$, respectively. The Conley-Zehnder indices of $\gamma_{R,N}$ and $\gamma_{D,N}$ are given by

$$\mu_{CZ}(\gamma_{R,N}) = 1 + 2\left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1} \right\rfloor, \quad \mu_{CZ}(\gamma_{D,N}) = 1 + 2\left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \right\rfloor$$

for $NS_R, NS_D \notin \mathbb{Z}\frac{2\pi}{(-2E)^{\frac{3}{2}}}$ where $S_R = \frac{2\pi}{(-2E)^{\frac{3}{2}}+1}$ and $S_D = \frac{2\pi}{(-2E)^{\frac{3}{2}}-1}$ are the periods of γ_R and γ_D, respectively. Moreover, the Conley-Zehnder index of $\gamma_{k,l}$ is $2k-1$ for each $k > l \geq 1$.

Let us explain the meaning of Proposition 7.1.2 and some related topic. From the above computation of Conley-Zehnder indices of all periodic orbits, in [7], they proved the dynamical convexity and therefore there exists a global disk-like surfaces of sections for each energy hypersurface of the rotating Kepler problem after the Levi-Civita transformation using the following remarkable statements in [28].

Definition 7.1.3 (Hofer-Wysocki-Zehnder)**.** Let $(\Sigma, \xi = \ker\lambda)$ be a contact 3-manifold. The contact form λ is called *dynamically convex* if $c_1(\xi)$ vanishes on $\pi_2(N)$ and $\mu_{CZ}(\gamma) \geq 3$ for any contractible periodic Reeb orbit γ.

Theorem 7.1.4 (Hofer-Wysocki-Zehnder)**.** *Let λ be a dynamically convex contact form on S^3. Then there exists a disk-like global surfaces of section for Reeb vector field.*

Because, in general, it is hard to know all Reeb orbits of a contact three manifold, it is hard to determine dynamical convexity using the definition. However, in [28], they gave a useful sufficient condition for dynamical convexity.

Theorem 7.1.5 (Hofer-Wysocki-Zehnder)**.** *A strictly convex regular energy hypersurface Σ of $\mathbb{R}^4$ with the canonical contact form λ_{can} is dynamically convex.*

As an application of above Theorems, one can see the following result for the restricted three body problem in [6].

Theorem 7.1.6 (Albers-Fish-Frauenfelder-Hofer-van Koert)**.** *Given $c > \frac{3}{2}$, there exists $\mu_0 = \mu_0(c) \in [0, 1)$ such that for all $\mu_0 < \mu < 1$ there exists a disk-like global surface of section for the hypersurface of the Levi-Civita regularized restricted three body problem of mass ratio μ with its Reeb vector field.*

In other words, they proved that for such pairs (μ, c), the Levi-Civita regularized energy hypersurfaces are strictly convex. On the other hand, in [7], they also disproved strict convexity for energy hypersurfaces of the rotating Kepler problem after Levi-Civita transformation. Thus Theorem 7.1.5 cannot be applied to the rotating Kepler problem. In this point of view, one can ask whether the rotating Kepler problem has the convex embedding or not. Recently, the answer was given in [24]. They proved that a combination of the Ligon-Schaaf and Levi-Civita regularizations provides strictly convex energy hypersurfaces of the regularized rotating Kepler problem in $\mathbb{R}^4$ below the critical level. This application of holomorphic curve theory to the celestial mechanics problem is well-organized in the up-coming book [23] of Frauenfelder and van Koert.

Now, we prove (2) and (3) of Theorem 7.1. We recall the anti-symplectic involutions

$$\mathcal{R}_1 : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2, \quad \mathcal{R}_2 : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$$

$$\mathcal{R}_1(q_1, q_2, p_1, p_2) = (q_1, -q_2, -p_1, p_2), \quad \mathcal{R}_2(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2)$$

of $T^*\mathbb{R}^2$. The Hamiltonian H_R of the rotating Kepler problem is invariant under these involutions, that is, $H_R = H_R \circ \mathcal{R}_i$ for $i = 1, 2$. A Hamiltonian chord from the Lagrangian $\mathrm{Fix}(\mathcal{R}_1)$ to itself corresponds to a symmetric orbit. We will compute the Robbin-Salamon indices of all Hamiltonian chords connecting $\mathrm{Fix}(\mathcal{R}_1)$ with itself in the rotating Kepler problem. In the rotating Kepler problem, every circular periodic orbit is symmetric with respect to the involution $\mathcal{R}_1$ and so we will call the Hamiltonian chord by the name of original periodic orbit like *retrograde chord, direct chord*. The retrograde and direct chords denoted by v_R^c and v_D^c, respectively. Note that a circular periodic orbit provide two Hamiltonian chords but we let any one of them be v_R(resp. v_D), then the other is $\mathcal{R}_{1*}v_R$(resp. $\mathcal{R}_{1*}v_D$). We have that

$$\gamma_R^c = v_R^c \# \mathcal{R}_{1*}v_R^c, \quad \gamma_D^c = v_D^c \# \mathcal{R}_{1*}v_D^c.$$

We have seen the k-fold covered ellipse in an l-fold covered coordinate system $\gamma_{k,l}$ form a torus family for each $c_{k,l}^- < c < c_{k,l}^+$. We have 4 discrete symmetric periodic orbits with respect to the involution $\mathcal{R}_1$. We will denote the corresponding Hamiltonian chord by $v_{k,l}$. We will denote by $v_{k,l}^c$ the $v_{k,l}$*-chord on the energy hypersurface* $H_R^{-1}(-c)$.

We will compute the indices of these chords v_R, v_D and $v_{k,l}$. First, we will determine the indices of the chords from circular orbits by direct computation. We introduce a polar coordinate $q_1 = r\cos\theta, \quad q_2 = r\sin\theta$ for q-coordinates. Consider the following transformation

$$p_1 = R\cos\theta - \frac{\Theta}{r}\sin\theta, \quad p_2 = R\sin\theta + \frac{\Theta}{r}\cos\theta,$$

then $T : (R, \Theta, r, \theta) \mapsto (q, p)$ is a symplectic transformation and, moreover, preserves the canonical 1-form, namely,

$$p_1 dq_1 + p_2 dq_2 = R dr + \Theta d\theta$$

holds. We define the transformed Hamiltonian

$$\tilde{H_R}(R, \Theta, r, \theta) := H_R \circ T(R, \Theta, r, \theta) = \frac{1}{2}(R^2 + \frac{\Theta^2}{r^2}) - \frac{1}{r} + \Theta$$

and compute its Hamiltonian vector field

$$X_{\tilde{H_R}} = \frac{\Theta^2 - r}{r^3}\frac{\partial}{\partial R} + R\frac{\partial}{\partial r} + (\frac{\Theta}{r^2} + 1)\frac{\partial}{\partial \theta}$$

We can compute the orbits for circular orbits. Since r is constant for circular orbits, we should have $R = 0$ and this implies $\Theta^2 = r$. Thus we can find the solution for orbit

$$\begin{pmatrix} R \\ \Theta \\ r \\ \theta \end{pmatrix}(t) = \begin{pmatrix} 0 \\ \pm\sqrt{r_0} \\ r_0 \\ \left(\frac{\pm 1}{r_0^{\frac{3}{2}}} + 1\right) t \end{pmatrix}$$

where $+$ is for the retrograde orbits and $-$ is for the direct orbits and so the *period* S_R *and* S_D *of the retrograde and direct orbit of radius* r_0 are given by

$$S_R = \frac{2\pi}{r_0^{-\frac{3}{2}} + 1} \quad \text{and} \quad S_D = \frac{2\pi}{r_0^{-\frac{3}{2}} - 1},$$

respectively. We shall look for the linearized Hamiltonian flow to compute the Robbin-Salamon indices. The differential of Hamiltonian vector field is

$$\begin{aligned} & dX_{\tilde{H}_R}\left(0, \Theta_0 = \pm\sqrt{r_0}, r_0, \left(\frac{\pm 1}{r_0^{\frac{3}{2}}} + 1\right) t\right) \\ = & \left(\frac{\partial}{\partial r}\right) dR + \left(\frac{2\Theta_0}{r_0^3}\frac{\partial}{\partial R} + \frac{1}{r_0^2}\frac{\partial}{\partial \theta}\right) d\Theta + \left(-\frac{1}{r_0^3}\frac{\partial}{\partial R} - \frac{2\Theta_0}{r_0^3}\frac{\partial}{\partial \theta}\right) dr \end{aligned}$$

along the circular orbit of radius r_0. Note that this is independent of t and we can write it as a matrix form

$$\left[dX_{\tilde{H}_R}\left(0, \Theta_0 = \pm\sqrt{r_0}, r_0, \left(\frac{\pm 1}{r_0^{\frac{3}{2}}} + 1\right) t\right)\right] = \begin{pmatrix} 0 & \frac{2\Theta_0}{r_0^3} & -\frac{1}{r_0^3} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r_0^2} & -\frac{2\Theta_0}{r_0^3} & 0 \end{pmatrix}$$

with respect to the ordered basis $\left\langle \frac{\partial}{\partial R}, \frac{\partial}{\partial \Theta}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right\rangle$. This basis gives a global trivialization for the phase space and this is a vertical preserving trivialization. Therefore, we can obtain the linearized Hamiltonian flow

$$\Psi_{\Theta_0,r_0}(t) := \left[d\phi^t_{\tilde{H}_R}(0,\Theta_0 = \pm\sqrt{r_0}, r_0, 0)\right] = \exp\left(t\begin{pmatrix} 0 & \frac{2\Theta_0}{r_0^3} & -\frac{1}{r_0^3} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r_0^2} & -\frac{2\Theta_0}{r_0^3} & 0 \end{pmatrix}\right)$$

by taking the exponential of the matrix $[dX_{\tilde{H}_R}]$. The Lagrangian $\mathcal{L}_1 = \mathrm{Fix}(\mathcal{R}_1)$ is given by

$$\mathcal{L}_1 = (q_1, 0, 0, p_2) \iff \mathcal{L}_1 = (0, \Theta, r, 0)$$

in (R, Θ, r, θ)-coordinate system. Therefore, the tangent space of $\mathcal{L}_1$ at $(0, \Theta_0, r_0, 0)$ is the space generated by $\frac{\partial}{\partial\Theta}$ and $\frac{\partial}{\partial r}$. We denote by $L_0 = \left\langle \frac{\partial}{\partial\Theta}, \frac{\partial}{\partial r} \right\rangle$. By Appendix A.2, under this set-up, we have to compute the following Robbin-Salamon index

$$\mu_{RS}(v) := \mu_{L_0}(\Psi_{\Theta_0,r_0}(t)L_0)$$

for the path of Lagrangians $\Psi_{\Theta_0,r_0}(t)L_0$ with respect to L_0. From a direct computation, we have the basis vectors

$$a_1(t) := \Psi_{\Theta_0,r_0}(t)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2\Theta_0}{r_0^{3/2}}\sin(\frac{t}{r_0^{3/2}}) \\ 1 \\ 2\Theta_0(1-\cos(\frac{t}{r_0^{3/2}})) \\ -\frac{3}{r_0^2}t + \frac{4}{r_0^{1/2}}\sin(\frac{t}{r_0^{3/2}}) \end{pmatrix},$$

$$a_2(t) := \Psi_{\Theta_0,r_0}(t)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{r_0^{3/2}}\sin(\frac{t}{r_0^{3/2}}) \\ 0 \\ \cos(\frac{t}{r_0^{3/2}}) \\ -\frac{2\Theta_0}{r_0^{3/2}}\sin(\frac{t}{r_0^{3/2}}) \end{pmatrix}$$

for the Lagrangian $\Psi_{\Theta_0,r_0}(t)L_0$. If $t \neq 0$, then $a_1(t) \notin L_0$. On the other hand, $a_2(t) \in L_0$ whenever $t = n\pi r_0^{\frac{3}{2}}$ for $n \in \mathbb{Z}$. Thus we have 2-dimensional crossing at $t = 0$ and 1-dimensional crossing at $t = n\pi r_0^{\frac{3}{2}}$ for $n \in \mathbb{Z}\backslash\{0\}$.

Following the definition in Appendix A.1, we compute the crossing forms at these times

$$Q(a_2(n\pi r_0^{\frac{3}{2}})) = \omega_0((0,0,(-1)^n,0),((-1)^{n+1}\frac{1}{r_0^3},0,0,(-1)^{n+1}\frac{2\Theta_0}{r_0^3})) = \frac{1}{r_0^3} > 0,$$

$$Q(a_1(0)) = \omega_0((0,1,0,0),(\frac{2\Theta_0}{r_0^3},0,0,\frac{4}{r_0^2})) = \frac{4}{r_0^2} > 0$$

where $\omega_0 = dR \wedge dr + d\Theta \wedge d\theta$ is the symplectic form and the row vectors are represented with respect to the ordered basis $\langle \frac{\partial}{\partial R}, \frac{\partial}{\partial \Theta}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle$.

Proposition 7.1.7. *We define by $v_{R,N}^c$ and $v_{D,N}^c$ the N-th concatenations of the retrograde chord v_R^c and the direct chord v_D^c from $\mathcal{L}_1$ to itself on $H_R^{-1}(-c)$, respectively. The Robbin-Salamon indices of $v_{R,N}$ and $v_{D,N}$ are given by*

$$\mu_{RS}(v_{R,N}^c) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1} \right\rfloor, \quad \mu_{RS}(v_{D,N}^c) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \right\rfloor$$

provided $\frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1}, \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \notin \mathbb{Z}$.

Proof. The time-lengths of the chords $v_{R,N}$ and $v_{D,N}$ are given by $\frac{N\pi}{r_0^{-\frac{3}{2}}+1}$ and $\frac{N\pi}{r_0^{-\frac{3}{2}}-1}$, respectively. The crossing appears for every $t = \pi r_0^{\frac{3}{2}}\mathbb{N}$ and each crossing has signature 1. Therefore, the Robbin-Salamon indicies are

$$\mu_{RS}(v_{R,N}) = \frac{1}{2}\text{sign } Q_0 + \sum_{0<t=k\pi r_0^{\frac{3}{2}} < \frac{N\pi}{r_0^{-\frac{3}{2}}+1}} \text{sign } Q_t,$$

$$\mu_{RS}(v_{D,N}) = \frac{1}{2}\text{sign } Q_0 + \sum_{0<t=k\pi r_0^{\frac{3}{2}} < \frac{N\pi}{r_0^{-\frac{3}{2}}-1}} \text{sign } Q_t,$$

if $\frac{N\pi}{r_0^{-\frac{3}{2}}+1}$ and $\frac{N\pi}{r_0^{-\frac{3}{2}}-1}$ are not integer multiples of $\pi r_0^{\frac{3}{2}}$. Thus, we have that $\mu_{RS}(v_{R,N}) = 1 + \left\lfloor \frac{N}{1+r_0^{\frac{3}{2}}} \right\rfloor$ and $\mu_{RS}(v_{D,N}) = 1 + \left\lfloor \frac{N}{1-r_0^{\frac{3}{2}}} \right\rfloor$ for the Robbin-Salamon indices when their radii are r_0 in the q-coordinates. Since $E = -\frac{1}{2r_0}$, we can rewrite the indices

$$\mu_{RS}(v^c_{R,N}) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1} \right\rfloor, \quad \mu_{RS}(v^c_{D,N}) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \right\rfloor$$

in terms of E. This proves Proposition 7.1.7. □

We have to determine the Maslov indices of the Hamiltonian chord $v^c_{k,l}$ corresponding to the k-fold covered ellipse in an l-fold covered coordinate system. However, it is indeed the analogue of the argument in [7] using local invariance of Morse homology.

Proposition 7.1.8. *The Robbin-Salamon index of $v_{k,l}$ is k.*

Proof. Fix $k > l \geq 1$. As the periodic orbit case, $v_{k,l}$ degenerates to $v^{c^+_{k,l}}_{D,k-l}$. For sufficiently small $\epsilon > 0$, we consider the indices of $\gamma^c_{D,k-l}$ for $c \in (c^+_{k,l} - \epsilon, c^+_{k,l} + \epsilon)$. Then $\mu_{RS}(\gamma^c_{D,k-l})$ changes from $k+1$ to k, when we increase c. Since $v_{k,l}$ exists for $c < c^+_{k,l}$ and the local homology should be preserved when c passes through $c^+_{k,l}$, we have that

$$\mu_{RS}(v_{k,l}) = k.$$

This proves Proposition 7.1.8. □

This proves (2) of Theorem 7.1.1. The proof of Theorem 7.1.1 (3) is completely analogous. We define the *retrograde chord w^c_R and the direct chord w^c_D from $\mathcal{L}_1 := Fix(\mathcal{R}_1)$ to $\mathcal{L}_2 := Fix(\mathcal{R}_2)$*. The *$N$-th concatenation of the chord is denoted by $w^c_{R,N}$ and $w^c_{D,N}$*, respectively. Note that first

$$\mathcal{L}_2 = (0, q_2, p_1, 0) \iff \mathcal{L}_2 = (0, \Theta, r, \frac{\pi}{2})$$

and so $L_0 = \langle \frac{\partial}{\partial \Theta}, \frac{\partial}{\partial r} \rangle$ becomes the tangent space of $\mathcal{L}_2$ in the trivialization as well. Second, we note that

$$w^c_{R,N} \# \mathcal{R}_{2*} w^c_{R,N} = v^c_{R,2N-1}, \quad w^c_{D,N} \# \mathcal{R}_{2*} w^c_{D,N} = v^c_{D,2N-1}. \tag{7.1.1}$$

From this, the number of crossing for $w_{R,N}$ and $w_{D,N}$ is given by

$$\left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}+1)} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}-1)} \right\rfloor,$$

respectively. We already showed that every crossing has signature 1. Therefore, we have proven Proposition 7.1.9.

Proposition 7.1.9. *We define by $w^c_{R,N}$ and $w^c_{D,N}$ the N-th concatenations of the retrograde chord w^c_R and the direct chord w^c_D from $\mathcal{L}_1$ to $\mathcal{L}_2$ on $H_R^{-1}(-c)$, respectively. The Robbin-Salamon indices of $w_{R,N}$ and $w_{D,N}$ are given by*

$$\mu_{RS}(w^c_{R,N}) = 1 + \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}+1)} \right\rfloor, \quad \mu_{RS}(w^c_{D,N}) = 1 + \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}-1)} \right\rfloor$$

provided $\frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}+1)}, \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}-1)} \notin \mathbb{Z}$.

We note that not every $\gamma_{k,l}$ gives the chord from $Fix(\mathcal{R}_1)$ to $Fix(\mathcal{R}_2)$. From the definition of $\gamma_{k,l}$, the period is $2\pi l$ and k-times iterates the ellipse in the inertial coordinate system. Among points on ellipse, only vertices on the major axis, say C the closest point and F the farthest point from the origin, can be points of $Fix(\mathcal{R}_1)$ or $Fix(\mathcal{R}_2)$. The shortest time from C to F or from F to C is $\frac{\pi l}{k}$. Thus $\frac{\pi l}{k}$ should divide $\frac{2\pi l}{4}$ in order to be possible to begin with C or F on $Fix(\mathcal{R}_1)$ and end with C or F on $Fix(\mathcal{R}_2)$. This implies $2|k$. As we discussed, $\gamma_{k,l}$ degenerates to $(k-l)$-iteration of γ_D and $w^c_{D,N}$ corresponds to $v^c_{D,2N-1}$ by (7.1.1). Therefore, $k-l$ must be an odd number. As the upshot of this argument, $\gamma_{k,l}$ can be a doubly symmetric periodic orbit only if $k > l \geq 1$ with $2|k, 2 \nmid l$. As an analogue of Proposition 7.1.8, we have the following Proposition.

Proposition 7.1.10. *The periodic $\gamma^c_{k,l}$ can be the chord $w^c_{k,l}$ if and only if $k > l \geq 1$ and $2|k, 2 \nmid l$. In this case , we have that*

$$\mu_{RS}(w_{k,l}) = \frac{k}{2}.$$

Proposition 7.1.9 and 7.1.10 completes the proof of Theorem 7.1.1 (3).

7.2 Spectrum of the rotating Kepler problem

Another important ingredient of symplectic homology and wrapped Floer homology is the action value of periodic Reeb orbits(resp. Reeb chords). Because the action values of Reeb chord are proportional to the action value of periodic Reeb orbits, for example, $\mathcal{A}(v_R) = \frac{1}{2}\mathcal{A}(\gamma_R)$, $\mathcal{A}(w_D) = \frac{1}{4}\mathcal{A}(\gamma_D)$ and so on. It suffices to compute the action values of periodic Reeb orbits.

One can compute the period of the closed Reeb orbit γ by the integration $\mathcal{A}(\gamma) = \int_\gamma \lambda$. We have seen that $(\Sigma_R^c, \lambda_{can})$ is a contact manifold for each $c > c_R^0 = \frac{3}{2}$. We will compute the period of every closed Reeb orbit in $(\Sigma_R^c, \lambda_{can})$.

First, we will compute the period of the retrograde circular orbit γ_R. Let r be the distance from the origin for the circular orbit of the Kepler problem. In q-coordinate, the circular orbit can be parametrized as follows

$$q_K^r(t) = (r\cos(\omega t), r\sin(\omega t)), \quad p_K^r(t) = \dot{q}_K^r(t) = (-r\omega\sin(\omega t), r\omega\cos(\omega t))$$

for some frequency ω. On the other hand, we have

$$\dot{p}_K^r(t) = (-r\omega^2\cos(\omega t), -r\omega^2\sin(\omega t)) = -\frac{q}{|q|^3} = (-r^{-2}\cos(\omega t), -r^{-2}\sin(\omega t))$$

from the Hamiltonian equation $\dot{p} = -\frac{\partial E}{\partial q}$. This implies $\omega = r^{-\frac{3}{2}}$. For this circular periodic orbit, the energy is given by $E = -\frac{1}{2r}$. The corresponding retrograde orbit in the rotating Kepler problem has the following parametrization

$$q_R^r(t) = (r\cos((r^{-\frac{3}{2}}+1)t), r\sin((r^{-\frac{3}{2}}+1)t))$$

on the q-coordinate. We have that

$$\begin{cases} \dot{q}_1 = p_1 - q_2 \\ \dot{q}_2 = p_2 + q_1 \end{cases} \implies \begin{cases} p_1 = \dot{q}_1 + q_2 \\ p_2 = \dot{q}_2 - q_1 \end{cases}$$

from the Hamiltonian equation $\dot{q} = \frac{\partial H_{RKP}}{\partial p}$. This implies

$$p_R^r(t) = (-r^{-\frac{1}{2}} \sin((r^{-\frac{3}{2}}+1)t), r^{-\frac{1}{2}} \cos((r^{-\frac{3}{2}}+1)t)).$$

Define $\gamma_R^r(t) := (q_R^r(t), p_R^r(t))$ and compute the integral

$$\begin{aligned} \mathcal{A}(\gamma_R^r) &= \int_{\Phi\circ\Psi(\gamma_R^r)} \lambda_{can} = \int_{\gamma_R^r} (\Phi\circ\Psi)^*\lambda_{can} \\ &= \int_{\gamma_R^r} -q dp = \int_0^{\frac{2\pi}{r^{-\frac{3}{2}}+1}} (r^{-1} + r^{\frac{1}{2}}) dt = 2\pi r^{\frac{1}{2}} \end{aligned}$$

where Φ, Ψ are symplectomorphisms defined in section 4.1. The energy $-c$ satisfies

$$-c = H_{RKP}(\gamma_R^r(t)) = E + L = -\frac{1}{2r} + r^{\frac{1}{2}} \iff c = \frac{1}{2r} - r^{\frac{1}{2}}$$

for this retrograde orbit in terms of r. In sum, the retrograde orbit γ_R^r of radius r satisfies that

$$\mathcal{A}(\gamma_R^r) = 2\pi r^{\frac{1}{2}}, \quad \gamma_R^r \subset H_{RKP}^{-1}\left(-\frac{1}{2r} + r^{\frac{1}{2}}\right).$$

Similarly we can compute the action and energy for the direct orbit of radius r. Let $\gamma_D^r(t) = (q_D^r(t), p_D^r(t))$ be the direct orbit of radius r in the rotating Kepler problem. Then we have

$$\begin{gathered} q_D^r(t) = (r\cos((-r^{-\frac{3}{2}}+1)t), r\sin((-r^{-\frac{3}{2}}+1)t)), \\ p_D^r(t) = (r^{-\frac{1}{2}} \sin((-r^{-\frac{3}{2}}+1)t), -r^{-\frac{1}{2}} \cos((-r^{-\frac{3}{2}}+1)t)) \end{gathered}$$

by similar computation in above. We can compute the action and the energy

$$\mathcal{A}(\gamma_D^r) = 2\pi r^{\frac{1}{2}}, \quad -c = H_{RKP}(\gamma_D^r(t)) = E + L = -\frac{1}{2r} - r^{\frac{1}{2}}$$

of γ_D^r with similar argument.

We want to express the action values of the retrograde and direct orbits in terms of L. In the retrograde orbit case, we have $L = r^{\frac{1}{2}}$ and we have

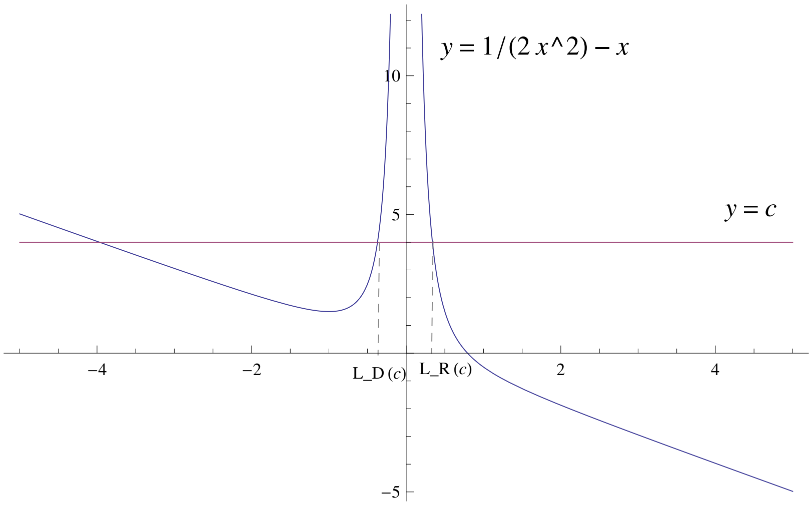

Figure 7.1: Definition of $L_R(c)$ and $L_D(c)$

$-c = E + L = -\frac{1}{2L^2} + L$. Therefore, the action of retrograde orbit γ_R^c on $H_{RKP}^{-1}(c)$ is given by

$$\mathcal{A}(\gamma_R^c) = 2\pi L_R(c)$$

where $L_R(c)$ is the positive zero of an equation $c = \frac{1}{2x^2} - x$ for x. In the direct orbit case, we have that $L = -r^{\frac{1}{2}}$. We also have $-c = E + L = -\frac{1}{2L^2} + L$ and $r < 1$. Therefore, the action of direct orbit γ_D^c on $H_{RKP}^{-1}(c)$ is given by

$$\mathcal{A}(\gamma_D^c) = -2\pi L_D(c)$$

where $-1 < L_D(c) < 0$ is the larger negative zero of the equation $c = \frac{1}{2x^2} - x$ for x. See Figure 7.2 for $L_R(c)$ and $L_D(c)$.

Finally, we have to compute the action of $\gamma_{k,l}$. We recall that $\gamma_{k,l}$ denotes a k-fold covered ellipse in an l-fold covered coordinate system and thus the period of $\gamma_{k,l}$ is $T_{k,l} = 2\pi l$ and the energy of the underlying ellipse is $E_{k,l} = -\frac{1}{2}(\frac{k}{l})^{\frac{2}{3}}$. In the Kepler problem, every simple periodic orbit in a fixed energy has the same action value because every simple periodic orbit corresponds to the great circle in the standard S^2 with round metric by Moser regularization. In fact, the action value of any simple periodic orbit γ^c on $E^{-1}(-c)$ is given by

$$\mathcal{A}_{KP}(\gamma^c) = 2\pi(2c)^{-\frac{1}{2}}$$

for each $c > 0$. Define

$$\lambda := (\Phi \circ \Psi)^* \lambda_{can} = -q dp$$

for the next computation. We compute the action of $\gamma_{k,l}$ as follows.

$$\begin{aligned}
\mathcal{A}(\gamma_{k,l}) &= \int_{\Phi\circ\Psi(\gamma_{k,l})} \lambda_{can} = \int_{\gamma_{k,l}} \lambda \\
&= \int_0^{T_{k,l}} \lambda(\dot{\gamma_{k,l}(t)}) dt \\
&= \int_0^{T_{k,l}} \lambda(X_{H_{RKP}}(\gamma_{k,l}(t))) dt \\
&= \int_0^{T_{k,l}} \lambda(X_E(\gamma_{k,l}(t)) + X_L(\gamma_{k,l}(t))) dt \\
&= \int_0^{T_{k,l}} \lambda(X_E(\gamma_{k,l}(t))) dt + \int_0^{T_{k,l}} \lambda(X_L(\gamma_{k,l}(t))) dt \\
&= k(2\pi(-2E_{k,l})^{-\frac{1}{2}}) + \int_0^{2\pi l} L(\gamma_{k,l}(t)) dt \\
&= 2\pi k \sqrt{\frac{1}{-2E_{k,l}}} + 2\pi l L
\end{aligned}$$

If we consider the periodic orbit $\gamma^c_{k,l}$ on $H^{-1}_{RKP}(-c)$, then we have $-c = E_{k,l} + L$ and thus we have

$$\mathcal{A}(\gamma^c_{k,l}) = 2\pi k \sqrt{\frac{1}{-2E_{k,l}}} + 2\pi l(-c - E_{k,l}) = 2\pi(-lc + \frac{3}{2} k^{\frac{2}{3}} l^{\frac{1}{3}})$$

for every $c \in (c^-_{k,l}, c^+_{k,l})$. We have seen that

$$c^-_{k,l} := -E_{k,l} - \sqrt{\frac{1}{-2E_{k,l}}}, \quad c^+_{k,l} := -E_{k,l} + \sqrt{\frac{1}{-2E_{k,l}}}$$

and so

$$c^-_{k,l} = \frac{1}{2}(\frac{k}{l})^{\frac{2}{3}} - (\frac{l}{k})^{\frac{1}{3}}, \quad c^+_{k,l} = \frac{1}{2}(\frac{k}{l})^{\frac{2}{3}} + (\frac{l}{k})^{\frac{1}{3}}.$$

This implies that

$$\frac{1}{2}(\frac{k}{l})^{\frac{2}{3}} - (\frac{l}{k})^{\frac{1}{3}} < c < \frac{1}{2}(\frac{k}{l})^{\frac{2}{3}} + (\frac{l}{k})^{\frac{1}{3}} \iff L_R(c)^3 < \frac{l}{k} < -L_D(c)^3$$

where $L_R(c) > 0$ and $-1 < L_D(c) < 0$ are zeros of $c = f(x) = \frac{1}{2x^2} - x$ see Figure 7.2 for the graph. We have proven the following Proposition.

Proposition 7.2.1. *Let $Spec(\Sigma^c_R, \lambda_{can})$ be the set of actions of the energy hypersurfaces of regularized the rotating Kepler problem at energy $-c$. Then we have*

$$\begin{aligned} Spec(\Sigma^c_R, \lambda_{can}) &= 2\pi L_R(c)\mathbb{N} \cup (-2\pi L_D(c))\mathbb{N} \\ &\cup \{2\pi(-lc + \frac{3}{2}k^{\frac{2}{3}}l^{\frac{1}{3}}) | \frac{l}{k} \in (L_R(c)^3, -L_D(c)^3), k > l \textit{ and } k, l \in \mathbb{N}\} \end{aligned}$$

for each $c > \frac{3}{2}$. The values $2\pi L_R(c)$ and $-2\pi L_D(c)$ are the actions of the retrograde and direct orbit, respectively, where

$$L_R(c) > 0 \textit{ and } -1 < L_D(c) < 0$$

are zeros of $c = f(x) = \frac{1}{2x^2} - x$.

We can have explicit formulas

$$L_R(c) = \frac{1}{2}\sqrt{\frac{3}{2c}} \sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2c}\right)^{\frac{3}{2}}\right)\right),$$
$$L_D(c) = \frac{1}{2}\sqrt{\frac{3}{2c}} \sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2c}\right)^{\frac{3}{2}}\right) + \frac{2\pi}{3}\right)$$

for the zeros of $c = \frac{1}{2x^2} - x$ using trigonometric identity, see Figure 7.2. As one can expect, it is easy to see that the retrograde orbit is the smallest action orbit in the rotating Kepler problem.

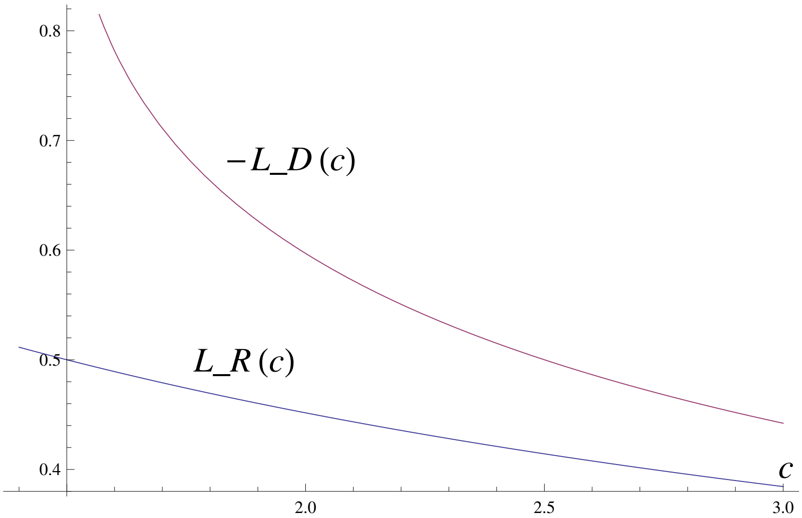

Figure 7.2: The graphs of $L_R(c)$ and $-L_D(c)$ with variable c

7.3 Computation of spectral invariant for the rotating Kepler problem

We will determine the chain complexes and the boundary maps for the symplectic homology of the Liouville domain M_R^c defined by the rotating Kepler problem.

The Conley-Zehnder indices of these orbits are

$$\mu_{CZ}(\gamma_{R,N}^c) = 1 + 2\left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}+1} \right\rfloor, \quad \mu_{CZ}(\gamma_{D,N}^c) = 1 + 2\left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}}-1} \right\rfloor$$

by Theorem 7.1.1. Using the relations $E_R(c) = -\frac{1}{2L_R(c)^2}$ and $E_D(c) = -\frac{1}{2L_D(c)^2}$, we define

$$\alpha_R(c) := \frac{(-2E_R(c))^{\frac{3}{2}}}{(-2E_R(c))^{\frac{3}{2}}+1} = \frac{1}{1+L_R(c)^3}, \quad \alpha_D(c) := \frac{(-2E_D(c))^{\frac{3}{2}}}{(-2E_D(c))^{\frac{3}{2}}-1} = \frac{1}{1+L_D(c)^3}$$

and we have the Conley-Zehnder indices

$$\mu_{CZ}(\gamma_{R,N}^c) = 1 + 2\lfloor N\alpha_R(c)\rfloor, \quad \mu_{CZ}(\gamma_{D,N}^c) = 1 + 2\lfloor N\alpha_D(c)\rfloor$$

of the Nth-iterated retrograde and direct orbit on energy $-c$ in terms of α_R, α_D. Note that $\alpha_R(c) < 1$ and $\alpha_D(c) > 1$ go to 1 as c goes to $+\infty$. For any integer $P \in \mathbb{N}$, we consider $c > c_R^0$ such that the inequalities

$$\frac{P}{P+1} < \alpha_R(c) < 1 \iff 0 < L_R(c)^3 < \frac{1}{P},$$
$$1 < \alpha_D(c) < \frac{P+1}{P} \iff 0 < -L_D(c)^3 < \frac{1}{P+1}$$

hold. Then we have

$$\mu_{CZ}(\gamma^c_{R,N}) = 2N - 1 \quad \text{for} \quad N = 1, 2, \cdots, P+1,$$
$$\mu_{CZ}(\gamma^c_{D,N}) = 2N + 1 \quad \text{for} \quad N = 1, 2, \cdots, P$$

for such c. This condition implies that there is no $\gamma^c_{k,l}$ satisfying $k \leq P+1$ for such c by Proposition 7.2.1. Then we have the periodic orbit $\gamma^c_{R,N}$(resp, $\gamma^c_{D,N}$) gives two generators, say $\overline{\gamma^c_{R,N}}, \underline{\gamma^c_{R,N}}$(resp, $\overline{\gamma^c_{D,N}}, \underline{\gamma^c_{D,N}}$), on the chain level by a suitable perturbation. The indices are $\mu_{CZ}(\overline{\gamma^c_{R,N}}) = 2N, \mu_{CZ}(\underline{\gamma^c_{R,N}}) = 2N-1$ and $\mu_{CZ}(\overline{\gamma^c_{D,N}}) = 2N+2, \mu_{CZ}(\underline{\gamma^c_{R,N}}) = 2N+1$. Thus we have

$$CF_*(K^b_{M^c_R}) = \begin{cases} \mathbb{Z}_2 & \text{if} \quad * = 0, 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if} \quad * = 2, 3, \cdots, 2P+1 \end{cases}$$

for any sufficiently large b. On the other hand, we recall from Example 5.1.12 that the symplectic homology

$$SH_*(M^c_R) = \begin{cases} \mathbb{Z}_2 & \text{if} \quad * = 0, 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if} \quad * = 2, 3, \cdots \end{cases}$$

of M^c_R. The boundary map in each grade less than $2P+2$ should be 0-map, because the number of generators in chain level coincides with the dimension of resulting homology in each grade less than $2P+2$. Moreover, we have two generators of constant orbits for a suitable Morse function inside Σ^c_R. Thus we have the following Theorem.

Theorem C1. Let P be a positive integer. If $c > \frac{P+3}{2(P+1)^{\frac{1}{3}}}$, then periodic orbits $\left\{\underline{\gamma^c_{R,N}}, \overline{\gamma^c_{R,N}}\right\}_{N=1,2,\cdots,P+1}$ and $\left\{\underline{\gamma_{D,N}}, \overline{\gamma_{D,N}}\right\}_{N=1,2,\cdots P}$ become generators of $SH_*(M^c_R)$. Moreover, if we define the homology classes $\Delta_{R,N} = \Psi^{-1}_{M^c_R}\left(\left[\underline{\gamma^c_{R,N}}\right]\right)$ for $N = 1, 2, \cdots, P+1$ and $\Delta_{D,N} = \Psi^{-1}_{M^c_R}\left(\left[\underline{\gamma^c_{D,N}}\right]\right)$ for $N = 1, 2, \cdots, P$ of $H_*(\Lambda S^2)$, then we have spectral invariants

$$c_{S^2}(M^c_R, \Delta_{R,N}) = 2\pi L_R(c)N, \quad \text{for } N = 1, \cdots P+1,$$

$$c_{S^2}(M^c_R, \Delta_{D,N}) = -2\pi L_D(c)N, \quad \text{for } N = 1, \cdots P$$

of $\Delta_{R,N}$ and $\Delta_{D,N}$ in the symplectic homology of M^c_R.

Proof. We have seen that if c satisfies $\frac{P}{P+1} < \alpha_R(c) < 1$ and $1 < \alpha_D(c) < \frac{P+1}{P}$, then $\left\{\underline{\gamma^c_{R,N}}, \overline{\gamma^c_{R,N}}\right\}_{N=1,2,\cdots,P+1}$ and $\left\{\underline{\gamma_{D,N}}, \overline{\gamma_{D,N}}\right\}_{N=1,2,\cdots P}$ with some constant classes generates $CF_k(K^b_{M^c_R})$ with $k = 0, 1, \cdots, 2P+1$ for sufficiently large b. For dimensional reason in the resulting homology $SH_*(M^c_R)$, the boundary map should be 0-map in $CF_k(K^b_{M^c_R})$ for $k = 0, 1, \cdots, 2P+1$. They become generators of $SH_*(M^c_R)$. Now we observe that the following equivalent condition

$$\begin{aligned} &\frac{P}{P+1} < \alpha_R(c) < 1 \quad \text{and} \quad 1 < \alpha_D(c) < \frac{P+1}{P} \\ \iff\ & L_R(c) < \frac{1}{P^{\frac{1}{3}}} \quad \text{and} \quad L_D(c) > \frac{-1}{(P+1)^{\frac{1}{3}}} \\ \iff\ & c > \max\left\{f\left(\frac{1}{P^{\frac{1}{3}}}\right), f\left(-\frac{1}{(P+1)^{\frac{1}{3}}}\right)\right\} = \frac{P+3}{2(P+1)^{\frac{1}{3}}} \end{aligned}$$

for c where $f(x) = \frac{1}{2x^2} - x$. This proves the first statement of Theorem. The second statement follows from the first statement and Proposition 7.2.1. This completes the proof of Theorem C1. □

Remark 7.3.1. If we consider $P = 1$, then c has to satisfy $-L_D(c)^3 < \frac{1}{2}$ and equivalently $c > 2^{\frac{2}{3}}$(the birth point of Hekuba orbit $\gamma_{2,1}$). This implies that if $c > 2^{\frac{2}{3}}$ then the retrograde and direct orbit are generators of $SH(M^c_R)$.

Now we consider the spectral invariant in wrapped Floer homology for M^c_R. Recall the notations and results in Example 5.2.9. Since M^c_R is invariant under the maps

$$\mathcal{R}_1 : T^*S^2 \to T^*S^2, \quad ((x_1, x_2, x_3), (y_1, y_2, y_3)) \mapsto ((-x_1, x_2, x_3), (y_1, -y_2, -y_3)),$$

$$\mathcal{R}_2 : T^*S^2 \to T^*S^2, \quad ((x_1, x_2, x_3), (y_1, y_2, y_3)) \mapsto ((x_1, -x_2, x_3), (-y_1, y_2, -y_3))$$

where $T^*S^2 = \{((x_1, x_2, x_3), (y_1, y_2, y_3)) \in T^*\mathbb{R}^3 ||x| = 1, x \cdot y = 0\}$. As we discussed in Example 5.2.3, the triple$(M^c_R, \lambda_{can}, \mathcal{R}_i)$, $i = 1, 2$ becomes a real Liouville domain. Moreover, the fixed point sets $Fix(\mathcal{R}_1)$ and $Fix(\mathcal{R}_2)$ are conormal bundles ν^*Q_1 and ν^*Q_2, respectively. We computed the wrapped Floer homology

$$WFH_*(\nu^*Q_1; M^c_R) = \begin{cases} \mathbb{Z}_2 & \text{for} \quad * = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * \geq 2. \end{cases}$$

in Example 5.2.9. By Theorem 7.1.1, we have that

$$\mu_{RS}(v^c_{R,N}) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}} + 1} \right\rfloor, \quad \mu_{RS}(v^c_{D,N}) = 1 + \left\lfloor \frac{N(-2E)^{\frac{3}{2}}}{(-2E)^{\frac{3}{2}} - 1} \right\rfloor.$$

Following the argument of the periodic orbit case, if $c > \frac{P+3}{2(p+1)^{\frac{1}{3}}}$, then

$$\begin{aligned} \mu_{RS}(v^c_{R,N}) &= N \quad \text{for} \quad N = 1, 2, \cdots, P+1, \\ \mu_{RS}(v^c_{D,N}) &= N + 1 \quad \text{for} \quad N = 1, 2, \cdots, P \end{aligned}$$

and $v^c_{k,l}$ does not appear for $k \leq P + 1$. Since the $\mathcal{R}_1$-action $\mathcal{R}_{1*}v$ of a Reeb chord v from ν^*Q_1 itself is a Reeb chord from ν^*Q_1 itself as well, the chain complex is given by

$$CF_*(\nu^*Q_1; K^b_{M^c_R}) = \begin{cases} \mathbb{Z}_2 & \text{for} \quad * = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 2, 3, \cdots P + 1. \end{cases}$$

including the constant chords for a suitable Morse function inside Σ_R^c. Thus the boundary map in each grade less than $P+2$ should be 0-map, because the number of generates in chain complex coincides with the dimension of resulting homology in each grade less than $P+2$. We have proven the following Theorem.

Theorem C2. Let P be a positive integer. If $c > \frac{P+3}{2(P+1)^{\frac{1}{3}}}$, then the Reeb chords $\{v_{R,N}^c, \mathcal{R}_{1*}v_{R,N}^c\}_{N=1,2,\cdots,P+1}$ and $\{v_{D,N}, \mathcal{R}_{1*}v_{D,N}\}_{N=1,2,\cdots P}$ become generators of $WFH_*(\nu^*Q_1; M_R^c)$. Moreover, if we define the homology classes $\Xi_{R,N} = \Psi_{M_R^c}^{-1}\left(\left[v_{R,N}^c\right]\right)$ for $N = 1, 2, \cdots, P+1$ and $\Xi_{D,N} = \Psi_{M_R^c}^{-1}\left(\left[v_{D,N}^c\right]\right)$ for $N = 1, 2, \cdots, P$ of $H_*(\mathcal{P}_{Q_1}S^2)$ where Q_1 is the equator of S^2 defined above, then we have the spectral invariants

$$c_{Q_1,S^2}(M_R^c, \Xi_{R,N}) = \pi L_R(c)N, \quad \text{for } N = 1, \cdots P+1,$$

$$c_{Q_1,S^2}(M_R^c, \Xi_{D,N}) = -\pi L_D(c)N, \quad \text{for } N = 1, \cdots P$$

of $\Xi_{R,N}$ and $\Xi_{D,N}$ in the wrapped Floer homology of M_R^c with respect to ν^*Q_1.

Now we consider chords from ν^*Q_1 to ν^*Q_2. We recall the wrapped Floer homology

$$WFH_*(\nu^*Q_1, \nu^*Q_2; M_R^c) = \begin{cases} \mathbb{Z}_2 & \text{for } \quad * = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } \quad * = 1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } \quad * \geq 2. \end{cases}$$

in Example 5.2.9. By Theorem 7.1.1, we have that

$$\mu_{RS}(w_{R,N}^c) = 1 + \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}+1)} \right\rfloor, \quad \mu_{RS}(w_{D,N}^c) = 1 + \left\lfloor \frac{(2N-1)(-2E)^{\frac{3}{2}}}{2((-2E)^{\frac{3}{2}}-1)} \right\rfloor.$$

If $c > 2^{\frac{2}{3}}$, then we have that

$$\frac{2}{3} < \alpha_R(c) < 1, \quad 1 < \alpha_D(c) < 2 \iff -L_D(c)^3 < \frac{1}{2}.$$

The chord $w_{2,1}$ does not exist by Proposition 7.2.1 and thus every $w^c_{k,l}$-chord has index greater than 1. Moreover, we have $\mu_{RS}(w^c_R) = 1, \mu_{RS}(w^c_D) = 1$ and $\mu_{RS}(w^c_{R,2}) = 2, \mu_{RS}(w^c_{D,2}) > 1$ if $c > 2^{\frac{2}{3}}$. Thus, including the constant Hamiltonian chord staying at the intersection points of Q_1 and Q_2, we have the chain complex

$$CF_*(\nu^*Q_1, \nu^*Q_2; K^b_{M^c_R}) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for} \quad * = 1 \end{cases}$$

for $* = 0, 1$. We denote by $w^{c,+}_R, w^{c,-}_R, w^{c,+}_D, w^{c,-}_D$ generators of $CF_1(\nu^*Q_1, \nu^*Q_2; K^b_{M^c_R})$. Because the dimension of resulting homology in this grade $* = 1$ is 3, there are three linearly independent generators defined as linear combinations of $w^{c,+}_R, w^{c,-}_R, w^{c,+}_D, w^{c,-}_D$. We prove the following Lemma.

Lemma 7.3.2. *Let $w^{c,+}_R, w^{c,-}_R, w^{c,+}_D, w^{c,-}_D$ be as above. Then there exists at least one nonzero homology class which can be expressed using only $w^{c,+}_R, w^{c,-}_R$. Moreover there exists at leas one nonzero homology class which cannot be expressed using only $w^{c,+}_R, w^{c,-}_R$.*

Proof. We choose a basis $\{A_k = [a_{k1}w^{c,+}_R + a_{k2}w^{c,-}_R + a_{k3}w^{c,+}_D + a_{k4}w^{c,-}_D] | k = 1, 2, 3, a_{ij} \in \mathbb{Z}_2\}$. If there exists k such that $a_{k3} = a_{k4} = 0$, then we are done. If there exist $k \neq l$ such that $a_{k3} = a_{l3}$ and $a_{k4} = a_{l4}$, then the homology class $A_k - A_l \neq 0$ can be represented by $w^{c,+}_R, w^{c,-}_R$ alone and so we are done. If above two cases do not happen, then the set of pairs $\{(a_{k3}, a_{k4})\}$ must be $\{(1, 0), (0, 1), (1, 1)\}$. Without loss of generality we assume that $(a_{13}, a_{14}) = (1, 0), (a_{23}, a_{24}) = (0, 1), (a_{33}, a_{34}) = (1, 1)$. Then the homology class $A_1 + A_2 - A_3 \neq 0$ can be represented by $w^{c,+}_R, w^{c,-}_R$. The second statement is trivial from dimension. This proves the Lemma. □

We can compute the spectral invariant of the classes, denoted by R and D, in Lemma 7.3.2.

Theorem C3. For $c > 2^{\frac{2}{3}}$, consider the homology classes R and D of $WFH_*(\nu^*Q_1, \nu^*Q_2; M^c_R)$ in Lemma 7.3.2. If we denote $\Pi_R = \Psi_{M^c_R}(R)$ and

$\Pi_D = \Psi_{M_R^c}(D)$ of $H_*(\mathcal{P}_{Q_1,Q_2}S^2)$, then we have the spectral invariants

$$c_{Q_1,Q_2,S^2}(M_R^c, \Pi_R) = \frac{1}{2}\pi L_R(c)N, \quad c_{Q_1,Q_2,S^2}(M_R^c, \Pi_D) = -\frac{1}{2}\pi L_D(c)N$$

of $\Pi_{R,N}$ and $\Pi_{D,N}$ in the wrapped Floer homology of M_R^c with respect to ν^*Q_1 and ν^*Q_2.

7.4 Inclusions between the rotating Kepler problem and Hill's lunar problem

In this section, we will discuss various inclusions between Liouville domains determined by the regularized energy hypersurfaces of the rotating Kepler problem and Hill's lunar problem. Recall the Hamiltonians (3.2.1) and (3.3.1). We have discussed in section 4.2 that they are fiberwise convex and define Liouville domains in T^*S^2 whenever energy levels below the critical level are fixed. We defined the Liouville domains M_R^c and $M_H^{c'}$ determined by the rotating Kepler problem and Hill's lunar problem for $c > c_R^0$ and $c' > c_H^0$, respectively. One can easily see the inclusions between different energy hypersurfaces of the same problem. For $c_1 > c_2 > c_R^0$ and $c_1' > c_2' > c_H^0$, we have

$$M_R^{c_1} \subset M_R^{c_2} \quad \text{and} \quad M_H^{c_1'} \subset M_H^{c_2'}$$

Namely, the energy hypersurface is getting smaller as the energy goes down. Now we want to know the inclusions between M_R^c and $M_H^{c'}$. We define increasing sequences

$$c_R^P := \frac{P+3}{2(P+1)^{\frac{1}{3}}}, \quad c_H^P := \frac{2P+8-\sqrt{(P+1)(P+9)}}{2(P+1)^{\frac{1}{3}}} \tag{7.4.1}$$

for $P = 1, 2, 3, \cdots$. We defined by $-c_R^0 = -\frac{3}{2}$ and $-c_H^0 = -\frac{3^{\frac{4}{3}}}{2}$ the critical values of the rotating Kepler problem and Hill's lunar problem. We state the main Theorem of this section.

Theorem B. For the fiberwise convex domains M_R^c and $M_H^{c'}$ in T^*S^2 defined by the regularized energy hypersurfaces of the rotating Kepler problem and Hill's lunar problem, we have the following inclusions in T^*S^2.
(1) $M_H^c \subset M_R^{c_R^1}$ for all $c \geq c_H^0$.
(2) $M_H^c \subset M_R^{c_R^P}$ if $c \geq c_H^P$ for all $P = 2, 3, 4, \cdots$.
(3) $M_R^{c+\frac{1}{2c^2}} \subset M_H^c$ for all $c > c_H^0$.

Let the map $\pi : T^*(\mathbb{R}^2 \setminus \{(0,0)\}) \to \mathbb{R}^2 \setminus \{(0,0)\}$, $\pi(q,p) = q$ be the obvious projection onto the q-coordinates. For $c > c_R^0$, we denote by $\mathfrak{R}_R^c := \overline{\bigcup_{d>c} \pi(H_R^{-1}(-d))^b}$ *Hill's region of the rotating Kepler problem of energy* $-c$ *near the origin*. Here, superscript b means the bounded component. Moreover, we denote by $\mathfrak{R}_R := \overline{\bigcup_{c>c_R^0} \mathfrak{R}_R^c}$ *Hill's region of the rotating Kepler problem near the origin*. We denote the Hill's regions of Hill's lunar problem by $\mathfrak{R}_H^c$ and $\mathfrak{R}_H$ similarly.

Lemma 7.4.1. *For $c > c_R^0$ and $c' > c_H^0$, the Hill's regions are given by*

$$\mathfrak{R}_R^c = \{(q_1, q_2) \in \mathbb{R}^2 | \frac{1}{|q|} + \frac{1}{2}|q|^2 < c, |q| < 1\},$$
$$\mathfrak{R}_H^{c'} = \{(q_1, q_2) \in \mathbb{R}^2 | \frac{1}{|q|} + \frac{3}{2}q_1^2 < c', |q_1| < 3^{\frac{-1}{3}}, |q_2| < 2 \cdot 3^{\frac{-4}{3}}\}.$$

Proof. See [32]. □

The figures of the Hill's regions for some energy levels are given in Figure 7.3 and Figure 7.4. We construct Proposition 7.4.2 in order to compare two Liouville domains.

Proposition 7.4.2. *We have the following criteria for inclusions.*

(1) $M_H^{c'} \subset M_R^c$ *if and only if* $H_R(q,p) + c \leq 0$ *for all* $(q,p) \in H_H^{-1}(-c')$ *with* $q \in \mathfrak{R}_H^{c'}$.

(2) $M_H^{c'} \subset M_R^c$ *if* $H_H(q,p) + c' \geq 0$ *for all* $(q,p) \in H_R^{-1}(-c)$ *with* $q \in \mathfrak{R}_R^c$.

(3) $M_R^c \subset M_H^{c'}$ *if and only if* $H_H(q,p) + c' \leq 0$ *for all* $(q,p) \in H_R^{-1}(-c')$ *with* $q \in \mathfrak{R}_R^c$

(4) $M_R^c \subset M_R^{c'}$ *if* $H_R(q,p) + c \geq 0$ *for all* $(q,p) \in H_H^{-1}(-c')$ *with* $q \in \mathfrak{R}_H^{c'}$.

for every $c > c_R^0$ *and* $c' > c_H^0$.

Proof. For a fixed $p \in \mathbb{R}^2$, we define the function $H_{R,p} : (\mathbb{R}^2 - (0,0)) \to \mathbb{R}$ by $H_{R,p}(q) := H_R(q,p)$. Then for any $c > c_R^0$, the curve $H_{R,p}^{-1}(-c)$ has one bounded component. We will denote this bounded component by $\sigma_{R,p}^c$. Since we know that the rotating Kepler problem is fiberwise convex, the closed curve $\sigma_{R,p}^c$ bounds a strictly convex domain, say $D_{R,p}^c$, containing the origin and $\sigma_{R,p}^c \subset \mathfrak{R}_R^c$ for all p. Following symplectomorphisms, $\Phi \circ \Psi(\sigma_{R,p}^c)$ becomes a fiber of Σ_R^c at p and thus $\Phi \circ \Psi(\sigma_{R,p}^c) \subset T_{\phi(p)}^* S^2$. We can define the fiber $\Phi \circ \Psi(\sigma_{H,p}^{c'})$ of $\Sigma_H^{c'}$ and the strictrly convex domain $D_{H,p}^{c'}$ enclosed by $\sigma_{H,p}^{c'}$ analogously.

Proof of (1): The inclusion $M_H^{c'} \subset M_R^c$ holds if and only if the fiber $\Phi \circ \Psi(\sigma_{H,p}^{c'})$ of $\Sigma_H^{c'}$ is contained inside the fiber $\Phi \circ \Psi(\sigma_{R,p}^c)$ of Σ_R^c for every $p \in \mathbb{R}^2$. Because inclusion relation is preserved by $\Phi \circ \Psi$, we have

$$M_H^{c'} \subset M_R^c \iff D_{H,p}^{c'} \subset D_{R,p}^c \iff \sigma_{H,p}^{c'} \subset D_{R,p}^c$$

for every $p \in \mathbb{R}^2$. Assume we have $\sigma_{H,p}^{c'} \subset D_{R,p}^c$, then the following holds

$$q \in \sigma_{H,p}^{c'} \implies H_{R,p}(q) \leq -c \implies H_R(q,p) + c \leq 0$$

for every $p \in \mathbb{R}^2$, because $H_{R,p}$ is less than $-c$ on $D_{R,p}^c$. Since we know that $\sigma_{H,p}^{c'}$ is a closed curve including the origin on its inside, if the inequality $H_R(q,p) + c \leq 0$ holds for every $q \in \sigma_{H,p}^{c'}$, then q has to on $\mathfrak{R}_R^c$. Therefore, the converse is also true. This proves (1).

Proof of (2): By similar argument in the proof of (1), we have

$$\begin{aligned} M_H^{c'} \subset M_R^c &\iff \sigma_{R,p}^c \subset \mathbb{R}^2 \backslash D_{H,p}^{c'} \text{ for all } p \\ &\Longleftarrow H_{H,p}(q) \geq -c' \text{ for every } q \in \sigma_{R,p}^c \text{ for all } p \\ &\iff H_H(q,p) + c' \geq 0 \text{ for all } (q,p) \in H_R^{-1}(-c) \text{ such that } q \in \mathfrak{R}_R^c \end{aligned}$$

This proves (2).

(3) and (4) can be proven analogously. This proves Proposition 7.4.2.

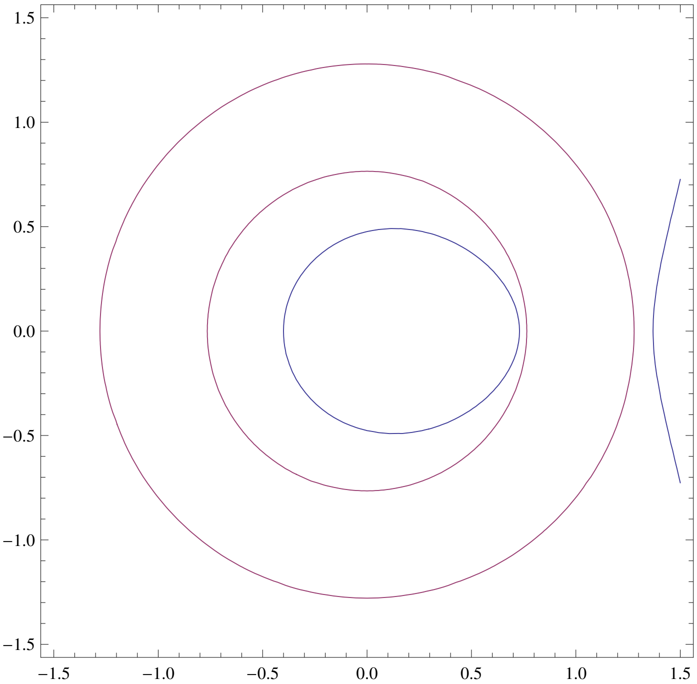

Figure 7.3: $\mathfrak{R}_R^{1.6}$ and $\sigma_{R,(0,1)}^{1.6}$

$\square$

Using Proposition 7.4.2, we will prove the Theorem B. First, we observe the following Theorem.

Theorem 7.4.3. *Every energy hypersurface of the regularized Hill's lunar problem below the critical value can be embedded in* $M_R^{2^{\frac{2}{3}}}$.

Proof. It is enough to show that $\Sigma_H^{c_H^0} \subset M_R^{2^{\frac{2}{3}}}$. Assume that $\Phi \circ \Psi(\bar{q}, \bar{p}) \in \Sigma_H^{c_H^0}$ and so

$$\frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} - \bar{q_1}^2 + \frac{1}{2}\bar{q_2}^2 + c_H^0 = 0, \quad \bar{q} \in \mathfrak{R}_H.$$

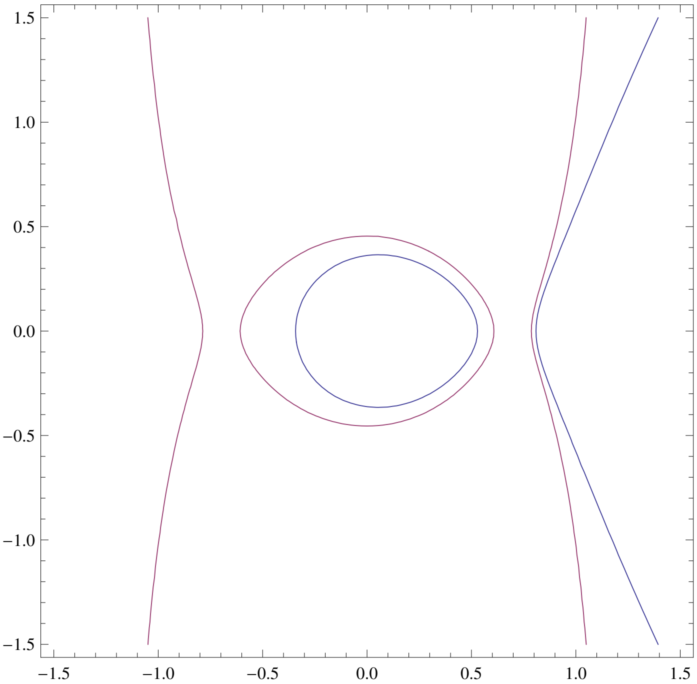

Figure 7.4: $\mathfrak{R}_H^{2.2}$ and $\sigma_{H,(0,1)}^{2.2}$

We compute the value of $H_R^{2^{\frac{2}{3}}} := H_R + 2^{\frac{2}{3}}$. Then we have

$$\begin{aligned} H_R^{2^{\frac{2}{3}}}(\bar{q},\bar{p}) &= \frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} + 2^{\frac{2}{3}} \\ &= \bar{q_1}^2 - \frac{1}{2}\bar{q_2}^2 - c_H^0 + 2^{\frac{3}{2}} \\ &\leq \bar{q_1}^2 + 2^{\frac{2}{3}} - c_H^0 \\ &\leq 3^{-\frac{2}{3}} + 2^{\frac{2}{3}} - \frac{3^{\frac{4}{3}}}{2} < 0. \end{aligned}$$

The last $\leq$ holds because $(\bar{q},\bar{p}) \in \mathfrak{R}_H$. Above inequality implies that $(\bar{q},\bar{p}) \in M_R^{2^{\frac{2}{3}}}$ by Proposition 7.4.2. This completes the proof of Theorem 7.4.3. $\square$

Recall that the sequences (7.4.1). Since $c_R^1 = 2^{\frac{2}{3}}$, Theorem 7.4.3 proves Theorem B (1). We note that the energy level $-2^{\frac{2}{3}}$ is the bifurcating point of Hekuba orbit: $\gamma_{2,1}$. This fact is important because it is hard to say about generators of symplectic homology and wrapped Floer homology when the energy level is between the critical value c_R^0 and $-2^{\frac{2}{3}}$ as we discussed in section

7.3. From the computation in the proof of Theorem, one can immediately see the following Corollary.

Corollary 7.4.4. *We have the embedding*

$$M_H^c \subset M_R^{c-3^{-\frac{2}{3}}}$$

for any $c > c_H^0$.

Theorem 7.4.5. *For the constants c_R^P and c_H^P in sequences (7.4.1), we have the following inclusion*

$$M_H^{c_H^P} \subset M_R^{c_R^P}$$

for each $P \in \{2, 3, 4, \cdots\}$.

Proof. It is enough to show that $\Sigma_H^{c_H^P} \subset M_R^{c_R^P}$. Assume that $\Phi \circ \Psi(\bar{q}, \bar{p}) \in \Sigma_H^{c_H^P}$. That is,

$$\frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} - \bar{q_1}^2 + \frac{1}{2}\bar{q_2}^2 + c_H^P = 0, \quad \bar{q} \in \mathfrak{R}_H^{c_H^P}.$$

We insert $(\bar{q}, \bar{p})$ in $H_R^{c_R^P} := H_R + c_R^P$, then we have

$$\begin{aligned} H_R^{c_R^P}(\bar{q}, \bar{p}) &= \frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} + c_R^P \\ &= \bar{q_1}^2 - \frac{1}{2}\bar{q_2}^2 - c_H^P + c_R^P \\ &\leq \bar{q_1}^2 + c_R^P - c_H^P \end{aligned}$$

and we want to prove the last term less that or equal to 0. It suffices to prove the following *Claim.*

Claim: If $\bar{q} \in \mathfrak{R}_H^{c_H^P}$, then $\bar{q_1}^2 \leq c_H^P - c_R^P$ for any $P \in \{2, 3, \cdots\}$.

Proof of Claim. For $\bar{q} \in \mathfrak{R}_H^{c_H^P}$, $|\bar{q_1}|$ attains its maximum, say $\bar{q_1}^M$, when $\bar{q_2} = 0$. It suffices to prove that

$$(\bar{q_1}^M)^2 \leq c_H^P - c_R^P$$

On the other hand, we know that $\bar{q_1}^M$ is the smaller positive zero of the equation

$$\frac{3}{2}x^2 + \frac{1}{x} = c_H^P = \frac{2P+8-\sqrt{(P+1)(P+9)}}{2(P+1)^{\frac{1}{3}}}$$

by Lemma 7.4.1. We solve the above equation and obtain

$$\bar{q_1}^M = \frac{\sqrt{P+9}-\sqrt{P+1}}{2(P+1)^{\frac{1}{6}}}$$

and so in fact we get $(\bar{q_1}^M)^2 = c_H^P - c_R^P$. This proves the *Claim.* $\square$

Claim implies that $H_R^{c_R^P}(\bar{q},\bar{p}) \leq 0$. By Proposition 7.4.2, this proves Theorem 7.4.5. $\square$

We have proven (2) of Theorem B. Since M_H^c shrinks as c increases, using Theorem 7.4.3 and Theorem 7.4.5, we formulate the inclusion for any $c > c_H^0$ in the following Corollary.

Corollary 7.4.6. *For any $c > c_H^0$, we have the following inclusions*

$$\begin{cases} M_H^c \subset M_R^{c_R^1} & \textit{if} \quad c \in (c_H^0, c_H^2), \\ M_H^c \subset M_R^{c_R^P} & \textit{if} \quad c \in [c_H^P, c_H^{P+1}) \textit{ for } P = 2,3,4,\cdots \end{cases}$$

We also have embeddings of opposite direction. Namely, the Liouville domain determined by the rotating Kepler problem can be embedded in the Liouville domain determined by Hill's lunar problem.

Proposition 7.4.7. *We have the embedding*

$$M_R^{c+\frac{1}{2c^2}} \subset M_H^c$$

for each $c > c_H^0$.

Proof. It suffices to prove that $\Sigma_R^{c+\frac{1}{2c^2}} \subset M_H^c$. Suppose that $(\bar{q},\bar{p}) \in \Sigma_R^{c+\frac{1}{2c^2}}$

and so

$$\frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} + c + \frac{1}{2c^2} = 0.$$

We evaluate $H_H^c := H_H + c$ at this $(\bar{q}, \bar{p})$. Then we have

$$\begin{aligned} H_H^c(\bar{q}, \bar{p}) &= \frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} - \bar{q_1}^2 + \frac{1}{2}\bar{q_2}^2 + c \\ &= -\frac{1}{2c^2} - \bar{q_1}^2 + \frac{1}{2}\bar{q_2}^2 \\ &\leq \frac{1}{2}\bar{q_2}^2 - \frac{1}{2c^2} \leq \frac{1}{2c^2} - \frac{1}{2c^2} = 0 \end{aligned}$$

The last inequality can be proven by the following *Claim*.

Claim: For any $(\bar{q}, \bar{p}) \in \Sigma_R^{c+\frac{1}{2c^2}}$, we have $|\bar{q}| \leq \frac{1}{c}$.

Proof of Claim. Since $(\bar{q}, \bar{p}) \in \Sigma_R^{c+\frac{1}{2c^2}}$, we have

$$\begin{aligned} &\frac{1}{2}|\bar{p}|^2 - \frac{1}{|\bar{q}|} + \bar{p_1}\bar{q_2} - \bar{p_2}\bar{q_1} + c + \frac{1}{2c^2} = 0 \\ \Rightarrow\ &\frac{1}{2}(\bar{p_1} + \bar{q_2})^2 + \frac{1}{2}(\bar{p_2} - \bar{q_1})^2 = \frac{1}{|\bar{q}|} + \frac{1}{2}|\bar{q}|^2 - c - \frac{1}{2c^2} \end{aligned}$$

This implies $\frac{1}{|\bar{q}|} + \frac{1}{2}|\bar{q}|^2 \geq c + \frac{1}{2c^2}$ and so we have $|\bar{q}| \leq \frac{1}{c}$. This proves the *Claim.* □

Therefore, we have that $\Sigma_R^{c+\frac{1}{2c^2}} \subset M_H^c$. By the criteria in Proposition 7.4.2, this proves Proposition 7.4.7. □

This proves (3) of Theorem B. We will use these inclusions to get estimates of action of Hill's lunar problem in the next section.

7.5 Estimates for spectral invariants of Hill's lunar problem

We combine the results in section 7.3 and 7.4 in this section. Using the properties of spectral invariants in symplectic homology and wrapped Floer homology discussed in chapter 6, we will get estimates for spectral invariants in symplectic homology and wrapped Floer homology of Liouville domains defined by Hill's lunar problem. Recall that we have defined the homology classes $\Delta_{R,N}$ and $\Delta_{D,N}$ of $H_*(\Lambda S^2)$ in Theorem C1. Using this homology classes, we prove the following Theorem.

Theorem 7.5.1. *For the homology classes $\Delta_R, \Delta_D \in H_*(\Lambda S^2)$, the inequalities*

$$c_{S^2}(M_H^c, \Delta_R) \geq 2\pi \frac{-1+\sqrt{1+8c^3}}{4c^2}, \quad c_{S^2}(M_H^c, \Delta_D) \geq 2\pi \frac{1+\sqrt{1+8c^3}}{4c^2}$$

hold for all $c > c_H^0 = \frac{3^{\frac{4}{3}}}{2}$.

Proof. By Theorem B (3), the inclusion $M_R^{c+\frac{1}{2c^2}} \subset M_H^c$ holds. Then we can deduce from Theorem A1 the inequalities

$$c_{S^2}(M_R^{c+\frac{1}{2c^2}}, \Delta_R) \leq c_{S^2}(M_H^c, \Delta_R), \quad c_{S^2}(M_R^{c+\frac{1}{2c^2}}, \Delta_D) \leq c_{S^2}(M_H^c, \Delta_D)$$

of spectral invariants for all $c > c_H^0$. Since $c + \frac{1}{2c^2} > 2^{\frac{2}{3}}$ for $c > c_H^0 = \frac{3^{\frac{4}{3}}}{2}$, the homology class $\Psi_{M_R^{c+\frac{1}{2c^2}}}(\Delta_R) = \Delta_R$ is represented by the retrograde orbit γ_R and same for the direct orbit. This implies that

$$c_{S^2}(M_R^{c+\frac{1}{2c^2}}, \Delta_R) = 2\pi L_R\left(c + \frac{1}{2c^2}\right), \quad c_{S^2}(M_R^{c+\frac{1}{2c^2}}, \Delta_D) = -2\pi L_D\left(c + \frac{1}{2c^2}\right)$$

by Theorem C1. Because $-1 < L_D\left(c + \frac{1}{2c^2}\right) < 0 < L_R\left(c + \frac{1}{2c^2}\right)$ are zeros of $c + \frac{1}{2c^2} = \frac{1}{2x^2} - x$, we have

$$L_R\left(c + \frac{1}{2c^2}\right) = \frac{-1+\sqrt{1+8c^3}}{4c^2}, \quad L_D\left(c + \frac{1}{2c^2}\right) = \frac{-1-\sqrt{1+8c^3}}{4c^2}$$

for all $c > c_H^0$. This completes the proof of Theorem 7.5.1. □

This provides us a simple and sharp lower bound for spectral invariants in symplectic homology of M_H^c. Let us discuss about upper bounds as well.

Theorem 7.5.2. *For the homology classes $\Delta_R, \Delta_D \in H_*(\Lambda S^2)$, the following inequalities*

$$\begin{aligned}
c_{S^2}(M_H^c, \Delta_R) &< 2\pi \times \frac{1}{2}\sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}}\sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})}\right)^{\frac{3}{2}}\right)\right)\\
&< 2^{-\frac{11}{6}}\cdot 3^{\frac{1}{2}}\pi\sec\left(\frac{1}{3}\arccos(2^{-\frac{5}{2}}\cdot 3^{\frac{3}{2}})\right) \approx 2\pi\times 0.490534\\
c_{S^2}(M_H^c, \Delta_D) &< -2\pi \times \frac{1}{2}\sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}}\sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})}\right)^{\frac{3}{2}}\right)+\frac{2\pi}{3}\right)\\
&< -2^{-\frac{11}{6}}\cdot 3^{\frac{1}{2}}\pi\sec\left(\frac{1}{3}\arccos(2^{-\frac{5}{2}}\cdot 3^{\frac{3}{2}})+\frac{2\pi}{3}\right) \approx 2\pi\times 0.793701
\end{aligned}$$

hold for all $c > c_H^0$.

Proof. By Theorem B (1) and Corollary 7.4.4, we know that $M_H^c \subset M_R^{c-3^{-\frac{2}{3}}} \subset M_R^{2^{\frac{2}{3}}+\epsilon}$ for sufficiently small $\epsilon > 0$ and, using monotonicity of c_{S^2}, we have the inequalities

$$\begin{aligned}
c_{S^2}(M_H^c, \Delta_R) &< c_{S^2}(M_R^{c-3^{-\frac{2}{3}}}, \Delta_R) = 2\pi L_R(c-3^{-\frac{2}{3}})\\
&\leq c_{S^2}(M_R^{2^{\frac{2}{3}}+\epsilon}, \Delta_R) = 2\pi L_R(2^{\frac{2}{3}}+\epsilon) < 2\pi L_R(2^{\frac{2}{3}}),\\
c_{S^2}(M_H^c, \Delta_D) &< c_{S^2}(M_R^{c-3^{-\frac{2}{3}}}, \Delta_D) = -2\pi L_D(c-3^{-\frac{2}{3}})\\
&\leq c_{S^2}(M_R^{2^{\frac{2}{3}}+\epsilon}, \Delta_D) = -2\pi L_D(2^{\frac{2}{3}}+\epsilon) < -2\pi L_D(2^{\frac{2}{3}})
\end{aligned}$$

for all $c > c_H^0$. Theorem follows by expressing L_R and L_D explicitly. □

From the above Theorem, there is an obvious Corollary. Let $l_1(\Sigma, \lambda)$ be the period of the shortest periodic Reeb orbit of contact manifold (Σ, λ). $l_1(\Sigma, \lambda)$ is called the *systole of the contact manifold* (Σ, λ).

Corollary 7.5.3. *We have the following estimates*

$$l_1(\Sigma_H^c, \lambda_{can}) < 2\pi \times \frac{1}{2}\sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}}\sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})}\right)^{\frac{3}{2}}\right)\right)$$

for the systole of the regularized Hill's lunar problem.

For example, if we consider $c = c_H^0$(in fact, arbitrarily close c to c_H^0), then we have spectral gap

$$2\pi \times 0.43029 \approx 2\pi \times \frac{-1+\sqrt{82}}{3^{\frac{8}{3}}} < c_{S^2}(M_H^{\frac{3^{\frac{4}{3}}}{2}}, \Delta_R) < 2\pi \times 0.49053,$$
$$2\pi \times 0.53713 \approx 2\pi \times \frac{1+\sqrt{82}}{3^{\frac{8}{3}}} < c_{S^2}(M_H^{\frac{3^{\frac{4}{3}}}{2}}, \Delta_D) < 2\pi \times 0.79370$$

for the contact manifold $(\Sigma_H^{c_H^0}, \lambda_{can})$. This means the spectrum of the contact manifold $(\Sigma_H^{c_H^0}, \lambda_{can})$ satisfies

$$Spec(\Sigma_H^{c_H^0}, \lambda_{can}) \cap (2\pi \times 0.43029, 2\pi \times 0.49053) \neq \phi,$$

$$Spec(\Sigma_H^{c_H^0}, \lambda_{can}) \cap (2\pi \times 0.53713, 2\pi \times 0.79370) \neq \phi.$$

Because the upper bound of these estimates holds for all $c > c_H^0 = \frac{3^{\frac{4}{3}}}{2}$, we can say $l_1(\Sigma_H^c, \lambda_{can}) < \pi$ for every $c > c_H^0$. We note that the equalities

$$L_R(c_R^P) = \frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}}, \quad L_D(c_R^P) = -(P+1)^{-\frac{1}{3}}$$

hold for all $P \in \mathbb{N}$. Following the proof of Theorem 7.5.2, we can prove the following Theorem using (2) of Theorem B.

Theorem 7.5.4. *Let P be a positive integer. Suppose that $c \in [c_H^P, c_H^{P+1})$. Then we have estimates of spectral invariants in symplectic homology of Hill's lunar problem*

$$2\pi N \frac{-1+\sqrt{1+8c^3}}{4c^2} \leq c_{S^2}(M_H^c, \Delta_{R,N}) \leq 2\pi N \frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}},$$

for all $N = 1, 2, \cdots, P+1$ and

$$2\pi N \frac{1+\sqrt{1+8c^3}}{4c^2} \leq c_{S^2}(M_H^c, \Delta_{D,N}) \leq 2\pi N (P+1)^{-\frac{1}{3}}$$

for all $N = 1, 2, \cdots, P$.

This provides spectral gaps for the contact manifold $(\Sigma_H^c, \lambda_{can})$. If $c \in$

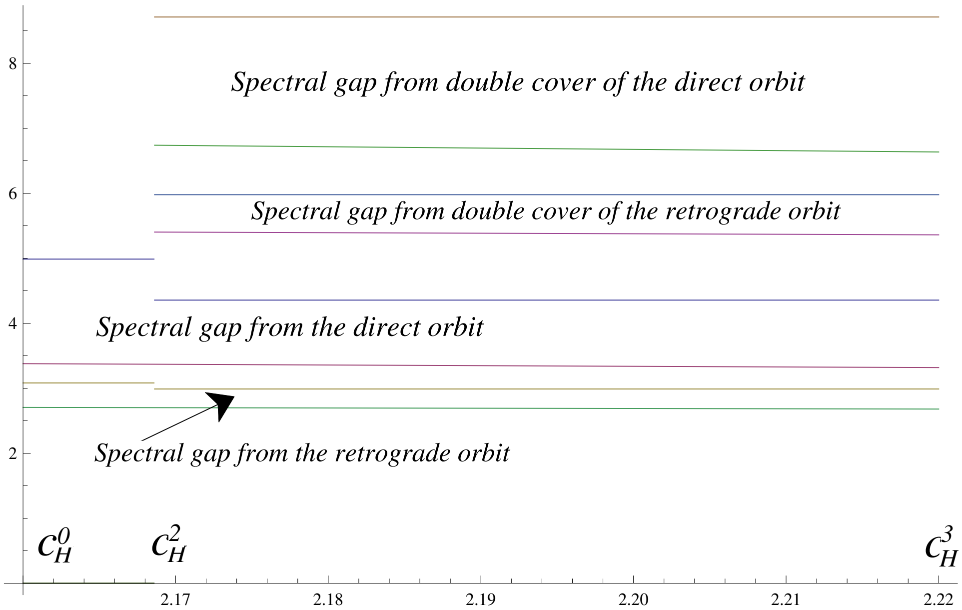

Figure 7.5: Estimates for the action of Hill's lunar problem on $c \in (c_H^0, c_H^3)$

$[c_H^P, c_H^{P+1})$, then we have

$$Spec(\Sigma_H^c, \lambda_{can}) \cap \left[2\pi N \frac{-1+\sqrt{1+8c^3}}{4c^2}, \quad 2\pi N \frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}} \right] \neq \phi$$

and

$$Spec(\Sigma_H^c, \lambda_{can}) \cap \left[2\pi N \frac{1+\sqrt{1+8c^3}}{4c^2}, \quad 2\pi N (P+1)^{-\frac{1}{3}} \right] \neq \phi$$

for $N = 1, 2, \cdots, P$ for any $P \in \mathbb{N}$. This implies only the existence of an orbit with an action range. Thus we do not know whether they are geometrically different or not. These spectral gaps are visualized in Figure 7.5 and 7.6. We can ensure the existence of at least one periodic orbit of action in these spectral gaps.

If we use Theorem C2 and Theorem B, then we can obtain analogue result for spectral invariants in wrapped Floer homology. Recall that we defined the equator $Q_1 = \{(x_1, x_2, x_3) \in S^2 | x_1 = 0\}$ of S^2 and $\mathcal{P}_{Q_1} S^2$ means the space of path from Q_1 to Q_1.

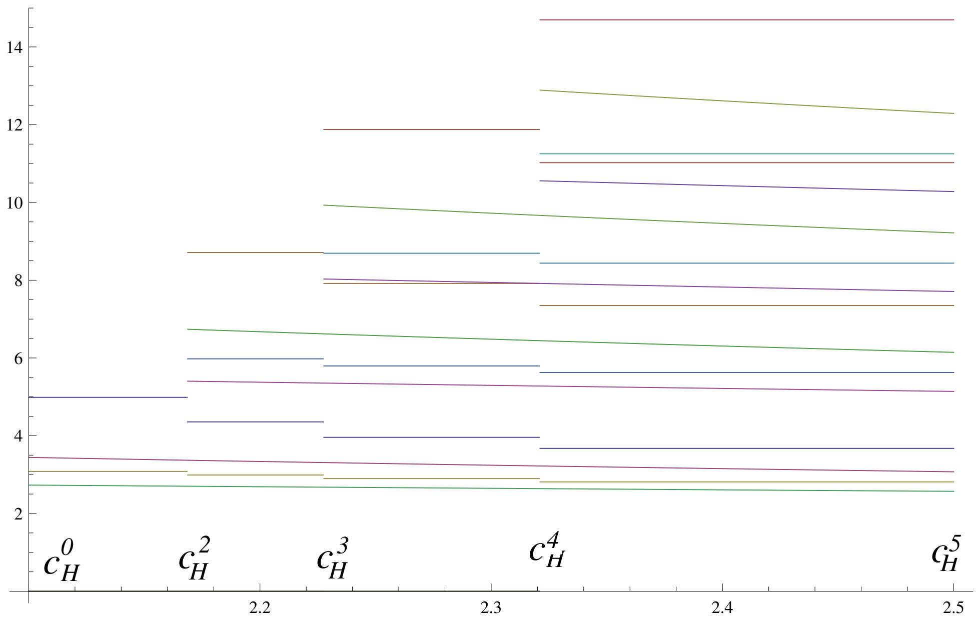

Figure 7.6: Estimates for the action of Hill's lunar problem on $c \in (c_H^0, c_H^5)$. Note that they can be overlapped from third cover.

Theorem 7.5.5. *For the homology classes $\Xi_R, \Xi_D \in H_*(\mathcal{P}_{Q_1}S^2)$, the inequalities*

$$c_{Q_1,S^2}(M_H^c, \Xi_R) \geq \pi\frac{-1+\sqrt{1+8c^3}}{4c^2}, \quad c_{Q_1,S^2}(M_H^c, \Xi_D) \geq \pi\frac{1+\sqrt{1+8c^3}}{4c^2}$$

hold for all $c > c_H^0 = \frac{3^{\frac{4}{3}}}{2}$.

Theorem 7.5.6. *For the homology classes $\Xi_R, \Xi_D \in H_*(\mathcal{P}_{Q_1}S^2)$, the follow-*

ing inequalities

$$\begin{aligned}
c_{Q_1,S^2}(M_H^c,\Xi_R) &< \pi\times\frac{1}{2}\sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}}\sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})}\right)^{\frac{3}{2}}\right)\right)\\
&< 2^{-\frac{17}{6}}\cdot 3^{\frac{1}{2}}\pi\sec\left(\frac{1}{3}\arccos(2^{-\frac{5}{2}}\cdot 3^{\frac{3}{2}})\right)\approx\pi\times 0.490534\\
c_{Q_1,S^2}(M_H^c,\Xi_D) &< -\pi\times\frac{1}{2}\sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}}\sec\left(\frac{1}{3}\arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})}\right)^{\frac{3}{2}}\right)+\frac{2\pi}{3}\right)\\
&< -2^{-\frac{17}{6}}\cdot 3^{\frac{1}{2}}\pi\sec\left(\frac{1}{3}\arccos(2^{-\frac{5}{2}}\cdot 3^{\frac{3}{2}})+\frac{2\pi}{3}\right)\approx\pi\times 0.793701
\end{aligned}$$

hold for all $c > c_H^0$.

Theorem 7.5.7. *Let P be a positive integer. Suppose that $c \in [c_H^P, c_H^{P+1})$. Then we have estimates of spectral invariants in wrapped Floer homology of Hill's lunar problem*

$$\pi N\frac{-1+\sqrt{1+8c^3}}{4c^2}\le c_{Q_1,S^2}(M_H^c,\Xi_{R,N})\le\pi N\frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}}$$

for $N = 1,2,\cdots,P+1$ and

$$\pi N\frac{1+\sqrt{1+8c^3}}{4c^2}\le c_{Q_1,S^2}(M_H^c,\Xi_{D,N})\le\pi N(P+1)^{-\frac{1}{3}}$$

for all $N = 1,2,\cdots,P$.

Let us explain the meaning of these analogous results. As we explained in Example 5.2.3, the conormal bundle ν^*Q_1 is the fixed point set of anti-symplectic involution $\mathcal{R}_1$. The anti-symplectic maps $\mathcal{R}_1$ and $\mathcal{R}_2$ on T^*S^2 in Example 5.2.3 correspond to the anti-symplectic maps $\mathcal{R}_1$ and $\mathcal{R}_2$ on $T^*\mathbb{R}^2$ in Remark 3.3.1 via stereographic projection Φ in (2.3.1) and switching map Ψ in (4.1.1). Thus, symmetric orbits with respect to $\mathcal{R}_1$ in T^*S^2 correspond to symmetric orbits with respect to $\mathcal{R}_1$ in $T^*\mathbb{R}^2$. Let $(\Sigma,\lambda,\mathcal{R})$ be a real contact manifold. We define the *set*

$$Spec^{\mathcal{R}}(\Sigma, \lambda) \subset Spec(\Sigma, \lambda)$$

of all periods of $\mathcal{R}$-symmetric periodic Reeb orbits. Then above Theorems about spectral invariants in $WFH_*(\nu^*Q_1; M_H^c)$ implies that if $c \in [c_H^P, c_H^{P+1})$, then we have

$$Spec^{\mathcal{R}_1}(\Sigma_H^c, \lambda_{can}) \cap \left[2\pi N \frac{-1+\sqrt{1+8c^3}}{4c^2}, \quad 2\pi N \frac{-(P+1)+\sqrt{(P+1)(P+9)}}{4(P+1)^{\frac{1}{3}}} \right] \neq \phi$$

and

$$Spec^{\mathcal{R}_1}(\Sigma_H^c, \lambda_{can}) \cap \left[2\pi N \frac{1+\sqrt{1+8c^3}}{4c^2}, \quad 2\pi N (P+1)^{-\frac{1}{3}} \right] \neq \phi$$

for $N = 1, 2, \cdots, P$ for any $P \in \mathbb{N}$. Thus the spectral gaps hold also for symmetric orbits.

Finally, we discuss the application of spectral invariants in wrapped Floer homology $WFH_*(\nu^*Q_1, \nu^*Q_2; M_R^c)$. Using Theorem C3 and Theorem B, we have the following Theorem.

Theorem 7.5.8. *If $c > c_H^0$, then we have estimates*

$$\begin{aligned} \pi \frac{-1+\sqrt{1+8c^2}}{8c^2} &< c_{Q_1,Q_2,S^2}(M_H^c, \Pi_R) \\ &< \pi \times \frac{1}{4} \sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}} \sec\left(\frac{1}{3} \arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})} \right)^{\frac{3}{2}} \right) \right) \end{aligned}$$

and

$$\begin{aligned} \pi \frac{1+\sqrt{1+8c^2}}{8c^2} &< c_{Q_1,Q_2,S^2}(M_H^c, \Pi_D) \\ &< \pi \times \frac{1}{4} \sqrt{\frac{3}{2(c-3^{-\frac{2}{3}})}} \sec\left(\frac{1}{3} \arccos\left(\left(\frac{3}{2(c-3^{-\frac{2}{3}})} \right)^{\frac{3}{2}} \right) + \frac{2\pi}{3} \right) \end{aligned}$$

for spectral invariants of $\Pi_R, \Pi_D \in H_*(\mathcal{P}_{Q_1,Q_2}S^2)$ *in wrapped Floer homology of* M_H^c *with respect to* ν^*Q_1 *and* ν^*Q_2.

As we discussed in Example 5.2.9, a chord from ν^*Q_1 to ν^*Q_2 corresponds to a doubly symmetric periodic orbit. Thus we can say that there exists a doubly symmetric orbit whose action is less than π for the regularized Hill's lunar problem of energy below the critical value.

Appendix A

Appendix: Maslov index

A.1 Appendix: Maslov indices for paths of Lagrangian subspaces

We summarize definitions and results in [46] for the definition of indices which are used as grades in symplectic homology and wrapped Floer homology in this thesis. We begin with the Maslov index for the loop. In order to define the Maslov index for the loop, we have to define the Lagrangian Grassmannian and research its topology.

For the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, a subspace L is called *Lagrangian subspace* if L is n-dimensional and $\omega_0(x_1, x_2) = 0$ for all $x_1, x_2 \in L$. We call the space of all Lagrangian subspace $\Lambda(n)$ by *Lagrangian Grassmannian*. Then it is easy to see that $\Lambda(n)$ has the structure of a homogeneous space

$$\Lambda(n) = U(n)/O(n)$$

because $U(n)$ acts on $\Lambda(n)$ transitively and $O(n)$ is the stabilizer. If we define the following map

$$\rho : U(n)/O(n) \to S^1, [A] \mapsto \det A^2,$$

then this is well-defined and gives an isomorphism for the fundamental group, see [10]. In other words, the induced homomorphism

$$\rho_* : \pi_1(\Lambda(n)) \to \pi_1(S^1) \cong \mathbb{Z}$$

is an isomorphism and so $\pi_1(\Lambda(n)) \cong \mathbb{Z}$. For a loop $\lambda : S^1 \to \Lambda(n)$ in the Lagrangian Grassmannian, we define *Maslov index of* λ by $\rho_*[\lambda] \in \mathbb{Z}$. Explicitly, this is the degree of the map

$$\rho \circ \lambda : S^1 \to S^1$$

for the loop λ on $\Lambda(n)$.

We want to generalize the Maslov index for loops to the Maslov index for paths. For this generalization, we need to define the Maslov cycle defined in [10]. We fix a base point $L_0 \in \Lambda(n)$ and define the following submanifold of $\Lambda(n)$

$$\Lambda^k(n) := \{L \in \Lambda(n) | \dim(L \cap L_0) = k\}$$

for each $k = 0, 1, \cdots, n$. Then we state the following facts as Proposition.

Proposition A.1.1. *The spaces have the following dimensions.*
(1) $\dim \Lambda(n) = \frac{n(n+1)}{2}$*,*
(2) $\dim \Lambda^k(n) = \frac{n(n+1)}{2} - \frac{k(k+1)}{2}$*.*
In particular, $\Lambda^0(n)$ *is an open subset of* $\Lambda(n)$ *and* $\Lambda^1(n)$ *has codimension 1 in* $\Lambda(n)$*.*

The closure of $\Lambda^1(n)$

$$\Lambda_{L_0} := \overline{\Lambda^1(n)} = \cup_{k=1}^{n} \Lambda^k(n)$$

is called *Maslov cycle (associated to* L_0*)*. The Maslov index for a loop can be interpreted as an intersection number with the Maslov cycle. For a loop of Lagrangian subspace, this intersection number is independent of the choice of base point L_0, see [10]. The advantage of this interpretation is allowing us to generalize to the path case. Robbin and Salamon proposed in [46] a Maslov index for any path regardless of where its endpoints lies. In order to define this index, we introduce the following identification of $T_{L_0}\Lambda(n)$ with quadratic form on L_0. We fix a Lagrangian complement L_1 of L_0, that is, $\mathbb{R}^{2n} = L_0 \oplus L_1$. Then we can define an isomorphism

$$T_{L_0}\Lambda(n) \to S^2(L_0), \quad \hat{L} \mapsto Q_{L_0}(\hat{L})$$

between the tangent space at L_0 and the space of quadratic forms. Let us define the quadratic form $Q_{L_0}(\hat{L})$ on L_0. For a path $L(t) \in \Lambda(n)$ such that $L(0) = L_0$ and $\frac{d}{dt}L(0) = \hat{L}$, we can find $w(t) \in L_1$ such that $v + w(t) \in L(t)$ for small t for each $v \in L_0$. Now we define that

$$Q_{L_0}(\hat{L})(v) := \left.\frac{d}{dt}\right|_{t=0} \omega_0(v, w(t)).$$

It is proved in [46] that $Q_{L_0}(\hat{L})$ defines a quadratic form on L_0 and is independent of the choice of L_1.

Definition A.1.2. We fix a base point $L \in \Lambda(n)$. Let $\lambda : [a, b] \to \Lambda(n)$ be smooth curve of Lagrangian subspaces. A number $t \in [a, b]$ is called a *crossing for* λ if $\lambda(t) \cap L \neq \{0\}$. At a crossing t, the quadratic form

$$\Gamma(\lambda, L; t) = Q_{\lambda(t)}(\dot{\lambda}(t))|_{\lambda(t)\cap L}$$

is called the *crossing form at* t. A crossing t is called *regular* if the corresponding crossing form is nonsingular.

For a path $\lambda : [a, b] \to \Lambda(n)$ with only regular crossings, we define the *Maslov index of* λ

$$\mu_L(\lambda) := \frac{1}{2} sign\Gamma(\lambda, L; a) + \sum_{a<t<b} sign\Gamma(\lambda, L; t) + \frac{1}{2} sign\Gamma(\lambda, L; b)$$

with respect to L. Using the following Lemma, we can extend this definition to paths having nonregular crossing.

Lemma A.1.3. *Every Lagrangian path is homotopic with fixed endpoints to a path having only regular crossings.*

We recall the following Theorem in [46].

Theorem A.1.4 (Robbin-Salamon)**.** *The Maslov index satisfies the following properties.*

- (**Naturality**) *For a symplectic matrix* Φ*,* $\mu_{\Phi L}(\Phi\lambda) = \mu_L(\lambda)$*.*
- (**Catenation**) *For* $a < c < b$*,* $\mu_L(\lambda) = \mu_L(\lambda|_{[a,c]}) + \mu_L(\lambda|_{[c,b]})$*.*
- (**Product**) $\mu_{L_0 \oplus L_1}(\lambda_0 \oplus \lambda_1) = \mu_{L_0}(\lambda_0) + \mu_{L_1}(\lambda_1)$*.*
- (**Localization**) *If* $L = \mathbb{R}^n \times \{0\}$ *and* $\lambda(t) = Graph(A(t))$ *for a path of symmetric matrices* $A(t) \in M_{n\times n}(\mathbb{R})$*, then* $\mu_L(\lambda) = \frac{1}{2} sign A(b) - \frac{1}{2} sign A(a)$*.*
- (**Homotopy**) *If two paths* $\lambda_0, \lambda_1 : [a,b] \to \Lambda(n)$ *with* $\lambda_0(a) = \lambda_1(a)$ *and* $\lambda_0(b) = \lambda_1(b)$ *are homotopic relative to endpoints if and only if they have the same Maslov index.*
- (**Zero**) *Every path* $\lambda : [a,b] \to \Lambda^k(n)$ *has Maslov index 0.*

We have defined the Maslov index for paths of Lagrangian submanifold. It is convenient to have such index for paths of symplectic matrices and it is called the Conley-Zehnder index. We can explain the Conley-Zehnder index in terms of the Maslov index discussed above.

We denote by $Sp(2n)$ the group of $2n \times 2n$ symplectic matrices. Define its subset $Sp^*(2n)$ of all $2n \times 2n$ symplectic matrices which do not have 1 as an eigenvalue. Note that $Sp^*(2n)$ is open and dense in $Sp(2n)$. We have a Maslov type index on the *set*

$$SP(2n) = \{\Psi : [0,1] \to Sp(2n) | \Psi(0) = I, \Psi(1) \in Sp^*(2n)\}$$

of paths in $Sp(2n)$. Consider the symplectic vector space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \Omega = (-\omega_0) \times \omega_0)$. For $\Psi \in SP(2n)$, it is easy to see $Graph(\Psi(t)) \subset (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \Omega)$ is a Lagrangian subspace for each $t \in [0,1]$. Thus we have a path of Lagrangian subspaces. We define the *Conley-Zehnder index*

$$\mu_{CZ}(\Psi) := \mu_{\triangle}(Graph(\Psi))$$

for $\Psi \in SP(2n)$ where $\triangle$ is the diagonal of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. We state properties of the Conley-Zehnder index for a path of symplectic matrices.

Theorem A.1.5. *For each* $n \in \mathbb{N}$, *there is a unique map*

$$\mu^n_{CZ} : SP(2n) \to \mathbb{Z}$$

satisfying the following properties.

- (**Naturality**) *For any path* $\Psi : [0,1] \to Sp(2n)$, $\mu_{CZ}(\Psi^{-1}\Phi\Psi) = \mu_{CZ}(\Phi)$
- (**Homotopy**) *If* Φ_1 *and* Φ_2 *are homotopic in* $SP(2n)$, *then* $\mu_{CZ}(\Phi_1) = \mu_{CZ}(\Phi_2)$.
- (**Zero**) *If* $\Phi(s)$ *has no eigenvalue on the unit circle for* $s > 0$, *then* $\mu_{CZ}(\Phi) = 0$.
- (**Product**) *For* $n_1 + n_2 = n$ *and* $\Phi_1 \in SP(2n_1), \Phi_2 \in SP(2n_2)$, *we can regard* $\Phi_1 \oplus \Phi_2$ *as an element of* $SP(2n)$. *Then* $\mu_{CZ}(\Phi_1 \oplus \Phi_2) = \mu_{CZ}(\Phi_1) + \mu_{CZ}(\Phi_2)$.
- (**Loop**) *If* $\Psi : [0,1] \to Sp(2n)$ *is a loop, then* $\mu_{CZ}(\Psi\Phi) = \mu_{CZ}(\Phi) + 2m(\Psi)$.
- (**Signature**) *If* S *is a symmetric* $2n \times 2n$ *matrix with* $||S||_{op} < 2\pi$ *and* $\Phi(t) = \exp tJ_0S$, *then* $\mu_{CZ}(\Phi) = \frac{1}{2}sign(S)$.

In fact, the map μ^n_{CZ} *is uniquely determined by* (**Homotopy**), (**Loop**) *and* (**Signature**) *properties.*

A.2 Appendix: Maslov indices on cotangent bundles

In this Appendix, we introduce the vertical preserving trivialization for the computation of Maslov index for Hamiltonian periodic orbits and Hamiltonian chords connecting conormal bundles in a cotangent bundle space. This is borrowed from [1] and [2]. Thus see [1] and [2] for details.

Let $\pi : T^*N \to N$ be the cotangent bundle over a smooth orientable n-manifold N. We define the *vertical subbundle of* TT^*N by taking the fiber

$$\ker d\pi(x) \subset T_x T^*N$$

at $x \in T^*N$. We denote by T^vT^*N the vertical subbundle of TT^*N. We consider the standard symplectic vector space $(\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n, \omega_0)$. We denote by λ_0 the vertical Lagrangian subspace. We introduce *vertical preserving symplectic trivialization* by the following Lemma.

Lemma A.2.1. *(1) Let $x : S^1 = \mathbb{R}/\mathbb{Z} \to T^*N$ be a loop in T^*N. Then the symplectic vector bundle (x^*TT^*N, ω_{can}) has a symplectic trivialization*

$$\Gamma : S^1 \times \mathbb{R}^{2n} \to x^*TT^*N$$

such that $\Gamma(t)(\lambda_0) = T^v_{x(t)}T^*N$ for all $t \in S^1$.
(2) Let $x : [0,1] \to T^*N$ be a path in T^*N. Then the symplectic vector bundle (x^*TT^*N, ω_{can}) has a symplectic trivialization

$$\Gamma : [0,1] \times \mathbb{R}^{2n} \to x^*TT^*N$$

such that $\Gamma(t)(\lambda_0) = T^v_{x(t)}T^*N$ for all $t \in [0,1]$.

One can find the proof of Lemma A.2.1 in Lemma 1.1 of [1]. Using vertical preserving symplectic trivializations, we will define the Maslov indices of nondegenerate Hamiltonian periodic orbits and nondegenerate Hamiltonian chords. First, we consider the periodic orbit case. Let $H : S^1 \times T^*N \to \mathbb{R}$ be a time-dependent 1-periodic Hamiltonian on T^*N. Let $x : S^1 \to T^*N$ be a solution of the Hamiltonian system $\dot{x}(t) = X^t_H(x(t))$ of H. We can choose a vertical preserving symplectic trivialization

$$\Gamma : S^1 \times \mathbb{R}^{2n} \to x^*TT^*N$$

by Lemma A.2.1. We define a path of symplectic matrices

$$\Phi^\Gamma_x(t) = \Gamma(t)^{-1} d\phi^t_H(x(0))\Gamma(0) \in Sp(2n), \quad t \in [0,1]$$

from the linearized Hamiltonian flow $d\phi_H^t$. If x is nondegenerate, then $\Phi_x^\Gamma \in SP(2n)$. We define the *Conley-Zehnder index*

$$\mu_{CZ}(x) := \mu_{CZ}(\Phi_x^\Gamma)$$

for the periodic orbit x. The trivialization Γ can be removed by Lemma A.2.3 below.

Remark A.2.2. In section 5.1, we defined Conley-Zehnder indices of Hamiltonian periodic orbits in a Liouville domain using filling disk. In the cotangent bundle of a smooth orientable manifold, $c_1(TT^*N)$ vanishes on $\pi_2(T^*N)$. Therefore, the index does not depend on the choice of filling disk in the cotangent bundle case. Moreover, if x is a contractible periodic orbit, then the vertical preserving trivialization can be extended to the filling disk and the definitions coincide for contractible periodic orbits in T^*N.

We recall the conormal bundle

$$\nu^*Q := \{x \in T^*N | \pi(x) \in Q, x(v) = 0 \text{ for all } v \in T_{\pi(x)}Q\}$$

of a submanifold $Q \subset N$. Let Q_0 and Q_1 be submanifolds of N. Let $x : [0,1] \to T^*N$ be a Hamiltonian chord from ν^*Q_0 to ν^*Q_1, that is,

$$\dot{x}(t) = X_H^t(x(t)), \quad x(0) \in \nu^*Q_0, x(1) \in \nu^*Q_1.$$

We can choose a vertical preserving symplectic trivialization

$$\Gamma : [0,1] \times \mathbb{R}^{2n} \to x^*TT^*N$$

of x^*TT^*N by Lemma A.2.1. We define the subspaces V_0^Γ, V_1^Γ of $\mathbb{R}^n$ by

$$\nu^*V_0^\Gamma = \Gamma(0)^{-1}T_{x(0)}\nu^*Q_0, \quad \nu^*V_1^\Gamma = \Gamma(1)^{-1}T_{x(1)}\nu^*Q_1$$

where the conormal bundles ν^*V_i are taken from the identification $(\mathbb{R}^{2n}, \omega_0) \cong (T^*\mathbb{R}^n, \omega_{can})$. This trivialization can be achieved by considering canonical local trivializations along x, see [2]. Note that $\dim V_i^\Gamma = \dim Q_i$ for $i = 0, 1$.

We define the path of symplectic matrices

$$\Phi_x^\Gamma(t) = \Gamma(t)^{-1} d\phi_H^t(x(0))\Gamma(0) \in Sp(2n), \quad t \in [0,1].$$

This gives the following definition of *Maslov index*

$$\mu_{Q_0,Q_1}(x) := \mu_{\nu^* V_1^\Gamma}(\Phi_x^\Gamma \nu^* V_0^\Gamma) + \frac{1}{2}(\dim Q_0 + \dim Q_1 - \dim N)$$

of the Hamiltonian chord x. As before, this is independent of the choice of Γ by Lemma A.2.3.

Lemma A.2.3. *(1) If x is a Hamiltonian periodic orbit, then the Conley-Zehnder index $\mu_{CZ}(\Phi_x^\Gamma)$ does not depend on the choice of a vertical preserving symplectic trivialization Γ.*
(2) If x is a Hamiltonian chord connecting two conormal bundles $\nu^* Q_0$ and $\nu^* Q_1$, then the Robbin-Salamon index $\mu_{\nu^* V_0^\Gamma}(\Phi_x^\Gamma \nu^* V_0^\Gamma)$ does note depend on the choice of a vertical preserving symplectic trivialization Γ.

One can find the proof of Lemma A.2.3 (1) in Lemma 1.2 of [1] and (2) in Proposition 3.2 of [2].

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국문초록

본 학위논문에서는 향을 줄 수 있는 닫힌 다양체의 코탄젠트 번들에 포함된 파이버마다 별모양인 영역들(fiberwise star-shaped domains)의 플로어 호몰로지에서의 스펙트랄 불변량들(spectral invariant)을 사교기하적 용량(symplectic capacity)으로 재해석한다. 회전 케플러 문제로 정의되는 영역을 이차원 구의 코탄젠트 번들에서 정규화 한 파이버마다 별모양인 영역에 대한 스펙트랄 불변량들을 계산한다. 그리고 회전 케플러 문제로 정의되는 영역을 이차원 구의 코탄젠트 번들(T^*S^2)에서 정규화(regularization)한 영역들과 힐의 달 문제로 정의되는 영역을 이차원 구의 코탄젠트 번들에서 정규화한 영역들 사이에 포함관계를 증명한다. 회전 케플러 문제에서 스펙트랄 불변량 계산과 회전 케플로 문제와 힐의 달 문제 사이의 포함관계 증명을 결합하면, 스펙트랄 불변량의 단조성을 이용하여 힐의 달 문제에서 스펙트랄 불변량에 대한 추산값을 얻을 수 있다. 한 해밀턴 계에서 스펙트랄 불변량 값은 그 해밀턴 계가 가지는 주기적인 궤도의 주기에 해당하는 값을 갖게되는 성질을 증명 할 수 있다. 이 성질을 이용하여 앞서 언급한 힐의 달 문제에서 스펙트랄 불변량에 대한 추산값으로 주기적인 궤도의 주기에 대한 정보를 얻을 수 있다. 특히, 힐의 달 문제에 있는 가장 짧은 주기를 갖는 주기적인 궤도의 주기에 대한 상한값을 줄 수 있고 이 학위논문에서는 이 값이 π 보다 작다는 것을 입증한다. 라그랑지 경계 조건을 갖는 플로어 호몰로지를 고려하면 대칭인 주기적인 궤도에 대해서도 앞에서 언급한 추산값을 그대로 사용할 수 있고 라그랑지 경계 조건을 서로 다른 라그랑지 부분 다양체로 택하면 이중 대칭인 주기적인 궤도에 대한 가장 짧은 주기의 상한값 역시 π라는 것을 증명할 수 있다.

주요어휘: 해밀턴 동력학, 회전계 케플러 문제, 힐의 달 문제, 플로어 호몰로지, 스펙트랄 불변량, 파이버별 볼록성
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